ON THE MINIMIZATION OF MULTINOMIAL TAILS AND THE GUPTA-NAGEL CONJECTURE

ABSTRACT

This paper is primarily concerned with the problem of minimizing the lower tail of the multinomial distribution. During the study of that specific problem, we have developed an approach which we believe to be general and useful for solving a wide class of similar problems, even involving multinomial probabilities represented by tails with "irregular shapes". Concerning the main problem, we provide a self-contained proof that the minimum of the multinomial lower tail is actually reached, as conjectured by Shanti S. Gupta and Klaus Nagel in 1967 within the framework of subset-selection problems, at the equal probability configuration, i.e., when the cell probabilities are equal to one another. We also point out some novel inequalities and general properties involving multinomial probabilities and multinomial coefficients.

Keywords: multinomial distribution, lower tail, best selection, subset selection, indifference-zone selection, multinomial coefficients, partitions of integer, Pascal triangle, Schur-convex functions

1. INTRODUCTION TO THE PROBLEM

An open problem in the theory of multinomial distribution is the minimization of its lower *tail*. Apparently, such a problem is of general interest, since multinomial tails occur in several situations, both in theoretical research and technological applications. A conjecture, proposed by Gupta and Nagel (1967, p. 9), suggests an answer to the above important issue, and it claims that the minimum of the *multinomial lower tail* (and, hence, the maximum of the complementary upper tail) is obtained when the cell probabilities are equal to one another.

Gupta-Nagel Conjecture (about the minimum of the lower tail of a multinomial distribution).

Given the integers k, n, r, where $k \geq 2,\ 0 \leq r < n, n > 0$, the function:

$$\tau_r(\theta_1,\dots,\theta_k) \equiv \sum_{\substack{n_1 \geq 0,\dots,n_k \geq 0 \\ \sum_{j=1}^{k} n_j = n \\ n_1 \leq \min_{j=2,\dots,k} n_j + r}} \frac{n!}{n_1!\ \dots\ n_k!}\ \theta_1^{n_1}\ \dots\ \theta_k^{n_k}\ , \qquad [1.1]$$

where

$$0 < \theta_1 \leq \dots \leq \theta_k < 1\,,\ \sum_{j=1}^{k} \theta_j = 1 \qquad [1.2]$$

reaches the minimum at the *equal probability configuration*:

$$\theta_1 = \frac{1}{k},\ \dots\ ,\ \theta_k = \frac{1}{k}\ . \qquad [1.3]$$

Remarks
Due to the restriction

$$n_1 \le \min_{j=2,\dots,k} n_j + r\,, \qquad [1.4]$$

the function [1.1] is the *lower tail* of the multinomial distribution, and r is a parameter of *amplitude*.
For instance, in the *binomial* case (i.e., when $k = 2$), the above tail can be written as:

$$\tau_r(\theta_1, 1-\theta_1) \;=\; \sum_{n_1=0}^{[\frac{n+r}{2}]} \binom{n}{n_1} \theta_1^{n_1} (1-\theta_1)^{n-n_1}\,, \quad \theta_1 \in (0, \frac{1}{2}]\,. \qquad [1.5]$$

It is clear that all the complexity of the minimization problem arises from the constraint [1.4], which restricts the multinomial cumulative probability to be smaller than 1, and dependent on $\theta_1,\dots,\theta_k$. Also, we may note that, if it were $r \ge n$, the constraint [1.4] would be satisfied for any k, n, $n_1,\dots,n_k$ ($n_1 \ge 0,\dots,n_k \ge 0$, $\sum_{j=1}^{k} n_j = n$), and, therefore, $\tau_r(\theta_1,\dots,\theta_k) = 1$. Finally, observe that the assumption $\theta_1 \le \ \dots \ \le \theta_k$ can be, equivalently, replaced with $\theta_1 = \min\{\theta_1, \dots, \theta_k\}$, since the multinomial tail [1.1] is *symmetrical* with respect to $n_2, \dots, n_k$ and, hence, the order of $\theta_2, \dots, \theta_k$ is, actually, immaterial.

In order to provide an introductory discussion on the Gupta-Nagel conjecture, let us define some notation. We denote by

$$B_\theta(\alpha,\beta) \;\equiv\; \int_0^\theta t^{\alpha-1}\,(1-t)^{\beta-1}\,dt\,,\; 0 < \theta \le 1,\; \alpha > 0,\; \beta > 0\,, \text{ and} \qquad [1.6]$$

$$B(\alpha,\beta) \;\equiv\; B_1(\alpha,\beta)\,, \qquad [1.7]$$

the *incomplete Beta function* and the *Beta function*, respectively, and by

$$I_\theta(\alpha,\beta) \;\equiv\; \frac{B_\theta(\alpha,\beta)}{B(\alpha,\beta)} \qquad [1.8]$$

the *incomplete beta function ratio*, which is linked to the binomial upper tail by the following well-known relationships (easily obtainable by integrating the incomplete beta [1.6] by parts):

$$I_\theta(t,\, n-t+1) \;=\; \sum_{h=t}^{n} \binom{n}{h} \theta^h\, (1-\theta)^{n-h}, \quad 0 \le t \le n\,, \qquad [1.9]$$

$$I_\theta(\alpha,\beta) \;=\; 1 \;-\; I_{1-\theta}(\beta,\alpha)\;. \qquad [1.10]$$

We can observe that the *binomial* lower tail [1.5] can be written as:

$$\tau_r(\theta_1, 1-\theta_1) \;=\; 1 \;-\; \sum_{n_1=[\frac{n+r}{2}]+1}^{n} \binom{n}{n_1} \theta_1^{n_1}\,(1-\theta_1)^{n-n_1}\,, \quad \theta \in (0, \frac{1}{2}], \qquad [1.11]$$

by [1.9]-[1.10]

$$= \;1 \;-\; I_{\theta_1}([\frac{n+r}{2}]+1,\, n-[\frac{n+r}{2}]) \;=\; I_{1-\theta_1}(n-[\frac{n+r}{2}],\, [\frac{n+r}{2}]+1) \qquad [1.12]$$

by [1.6]-[1.8]

$$= \;\frac{1}{B(n-[\frac{n+r}{2}],\, [\frac{n+r}{2}]+1)} \int_0^{1-\theta_1} t^{n-[\frac{n+r}{2}]-1}\,(1-t)^{[\frac{n+r}{2}]}\,dt\,. \qquad [1.13]$$

The above integral representation of the binomial lower tail, shows that $\tau_r(\theta_1, 1-\theta_1)$ is minimized at $\theta_1 = 1/2$, and it might also suggest that, in order to solve the general conjecture, we could, as well, try to represent the *multinomial* lower tail [1.1] in terms of *Dirichlet integrals*. This can be done, for instance, through the well-known expansion of multinomial probabilities obtained by Olkin and Sobel (1965) and Stoka (1966) (cf. also, Olkin and Sobel (1972)), which is useful to recall here, as it provides insight on the nature of the difficulties in solving the Gupta-Nagel conjecture:

$$\sum_{n_1=0}^{s_1} \ldots \sum_{n_{k-1}=0}^{s_{k-1}} \frac{n!}{n_1! \ldots n_k!} \theta_1^{n_1} \ldots \theta_k^{n_k}$$

$$= 1 - \sum_{n_1=s_1+1}^{n} \ldots \sum_{n_{k-1}=s_{k-1}+1}^{n} \frac{n!}{n_1! \ldots n_k!} \theta_1^{n_1} \ldots \theta_k^{n_k} \qquad [1.14]$$

$$= 1 - \sum_{h=1}^{k-1} I_{\theta_h}(s_h + 1, n - s_h)$$
$$+ \sum_{h_1=0}^{k-1} \sum_{h_2=0,\, h_2>h_1}^{k-1} I_{\theta_{h_1},\theta_{h_2}}(s_{h_1} + 1, n - s_{h_1}; s_{h_2} + 1, n - s_{h_2})$$
$$- \ldots$$
$$+ (-1)^{k-1} I_{\theta_{h_1},\theta_{h_2},\ldots,\theta_{h_{k-1}}}(s_1 + 1, n - s_1; \ldots ; s_{k-1} + 1, n - s_{k-1}), \qquad [1.15]$$

where

$$I_{\theta_{h_1},\theta_{h_2},\ldots,\theta_{h_m}}(s_{h_1} + 1, n - s_{h_1}; \ldots ; s_{h_m} + 1, n - s_{h_m})$$
$$\equiv \frac{n!}{s_{h_1}! \ldots s_{h_m}! \, (n - m - \sum_{i=1}^{m} s_{h_i})!} \int_0^{\theta_{h_1}} \ldots \int_0^{\theta_{h_m}} t_1^{s_{h_1}} \ldots t_m^{s_{h_m}} (1 - \sum_{i=1}^{m} t_i)^{n-m-\sum_{i=1}^{m} s_{h_i}} dt_1 \ldots dt_m \quad . \qquad [1.16]$$

Given the structure of [1.15], minimizing a representation in terms of Dirichlet integrals of the multinomial tail [1.1], while it is trivial for the binomial case [1.13], was not possible for $k > 2$. Thus, a new approach appears necessary to solve the problem, and that is the object of the present article.

The structure of the paper is as follows. Section 2 contains preliminary results. Section 3 contains the main results, including a novel inequality and a proof of the Gupta-Nagel conjecture. Some discussion is made in Section 4, where it is pointed out how some of the methods introduced here can be applied to a large class of problems relevant to the minimization of multinomial probabilities. As an example, a further minimization problem is proposed. Statistical considerations, historical background and numerical illustrations are contained in sections 5, 6, 7, respectively.

2. PRELIMINARY RESULTS

Notation: ***two partitions of the sample space***

Given the integers *n* and *k*, $n > 0,\ k \geq 2$, we denote by:

$$S_{n,k} \equiv \left\{ (n_1,\ldots,n_k) \in Z^k \ \middle|\ \sum_{j=1}^{k} n_j = n \right\} \qquad [2.1]$$

the *sample space* of the multinomial distribution, i.e., the set of points having *k* nonnegative integral coordinates which sum to *n* (here *Z* denotes, as usual, the set of *nonnegative* integers).

Partition 1

For each integer $r,\ 0 \leq r < n$, consider the following partition of $S_{n,k}$:

$$S_{n,k} = C_{n,k,r} \cup W_{n,k,r} \qquad [2.2]$$

into two *disjoint* subsets, $C_{n,k,r}$ and $W_{n,k,r}$, $S_{n,k} = C_{n,k,r} \cup W_{n,k,r}$, $C_{n,k,r} \cap W_{n,k,r} = \varnothing$, defined as follows:

$$C_{n,k,r} \equiv \left\{ (n_1,\ldots,n_k) \in S_{n,k} \mid n_1 \le \min_{j=2,\ldots,k} n_j + r \right\} \qquad [2.3]$$

$$W_{n,k,r} \equiv \left\{ (n_1,\ldots,n_k) \in S_{n,k} \mid n_1 > \min_{j=2,\ldots,k} n_j + r \right\}. \qquad [2.4]$$

Note that the set $C_{n,k,r}$ is the part of $S_{n,k}$ over which the sum of the multinomial lower tail [1.1] is carried out, while $W_{n,k,r} = S_{n,k} - C_{n,k,r}$ is the complement of $C_{n,k,r}$ with respect to the sample space $S_{n,k}$. The reason for this notation is to comply with the meaning (cf. Section 5 on historical background) of the original problem dealt with by Gupta and Nagel (*subset-selection approach*), where these two sets represent the "*Correct Selection" zone* and "*Wrong Selection" zone*, respectively.

Partition 2

Also, let $P_0,\ldots,P_n$ be the partition of $S_{n,k}$ into $(n+1)$ *mutually disjoint* parts defined as:

$$S_{n,k} = \bigcup_{h=0}^{n} P_h , \qquad [2.5]$$

$$P_h \equiv \left\{ (n_1,\ldots,n_k) \in S_{n,k} \mid n_1 = h \right\}, \quad h = 0,\ldots,n . \qquad [2.6]$$

(This partition, which separates the configurations with different values of n_1, will be useful to express the multinomial tail in terms of binomial probabilities.)

Results (properties of the partitions)

The properties, of the two above partitions [2.2] and [2.5], which will be established in the following Lemmas 1 and 2, will be used in the next section to build a proof of the Gupta-Nagel conjecture.

Preliminary Remarks

First of all, note that a sequence $(P_0 , \ldots , P_j)$, $j \in \{0,\ldots,n-1\}$, in $C_{n,k,r}$ always exists, being formed at least by P_0 since $n_1 = 0$ satisfies $n_1 \le \min_{j=2,\ldots,k} n_j + r$.

Similarly, a sequence $(P_{i+1} , \ldots , P_n)$, $(i+1) \in \{1,\ldots,n\}$, in $W_{n,k,r}$ always exists, being formed at least by P_n, since $r < n$ by assumption, and $n_1 = n$ implies $\min_{j=2,\ldots,k} n_j = 0$.

Lemma 1.

Given *n* and *k*, $n > 0,\ k \ge 2$, for each r, $0 \le r < n$, it is possible to split the partition [2.5] into two *nonempty* sequences:

$$(P_0 \cup \ldots \cup P_j) \quad (P_{j+1} \cup \ldots \cup P_n), \qquad [2.7]$$

$j \in \left\{0,\ldots,n-1\right\}$, such that each part in the *first* sequence $(P_0 \cup \ldots \cup P_j)$ is a subset of $C_{n,k,r}$, i.e.,

$$\text{exists } j \text{ such that } \bigcup_{h=0}^{j} P_h \subset C_{n,k,r} . \qquad [2.8]$$

In particular, denoting by j_0 the *largest* integer in $\{0,\ldots,n-1\}$ such that $\bigcup_{h=0}^{j} P_h \subset C_{n,k,r}$ holds, while $P_h \cap W_{n,k,r} \ne \varnothing$ for any $h \in \{j_0+1, \ldots ,n\}$, we have:

$$j_0 = \begin{cases} = [\frac{n+r}{2}] & if \quad k = 2 \\ \\ = r & if \quad k \ge 3 \end{cases} \qquad [2.9]$$

In other words, we can say that $(P_0 \cup ... \cup P_{j_0})$ is the "longest" of the subsequences $(P_0 \cup ... \cup P_j)$ "fully contained" in $C_{n,k,r}$.

Lemma 2.
Given *n* and *k*, $n > 0,\ k \geq 2$, for each r, $0 \leq r < n$, it is possible to split the partition [2.5] into two *nonempty* sequences:

$$(P_0 \cup ... \cup P_i) \quad (P_{i+1} \cup ... \cup P_n), \qquad [2.10]$$

$(i+1) \in \{1,...,n\}$, such that each part in the *second* sequence $(P_{i+1} \cup ... \cup P_n)$ is a subset of $W_{n,k,r}$, i.e.,

$$\text{exists } i \text{ such that } \bigcup_{h=i+1}^{n} P_h \subset W_{n,k,r}. \qquad [2.11]$$

In particular, denoting by i_0 the *smallest* integer in $\{0,...,n-1\}$ such that $\bigcup_{h=i+1}^{n} P_h \subset W_{n,k,r}$ holds, while $P_h \cap C_{n,k,r} \neq \varnothing$ for any $h \in \{0,..., i_0\}$, we have:

$$i_0 = \begin{cases} = [\frac{n+r}{2}], \quad if\ k = 2 \\ = r \\ \quad if\ \ k \geq 3\ \ and\ (n < k) \cup (r > n-k) \\ = [\frac{n+r(k-1)}{k}]\ \ or\ (depending\ on\ n,\ k,\ r)\ = [\frac{n+r(k-1)}{k}] - 1 \\ \quad if\ \ k \geq 3\ \ and\ (n \geq k) \cap\ (0 \leq r \leq n-k) \end{cases} \qquad [2.12]$$

and i_0 is always between r and $n-k$, i.e.,

$$r \leq i_0 \leq n-k. \qquad [2.13]$$

In other words, we might say that $(P_{i_0+1}, ..., P_n)$ is the "longest" of the subsequences $(P_{i+1}, ..., P_n)$ "fully contained" in $W_{n,k,r}$.

Taking into account the symbolic representation provided by Figure 1, can be of help to the reader to visualize the intuitive meaning of the two lemmas.

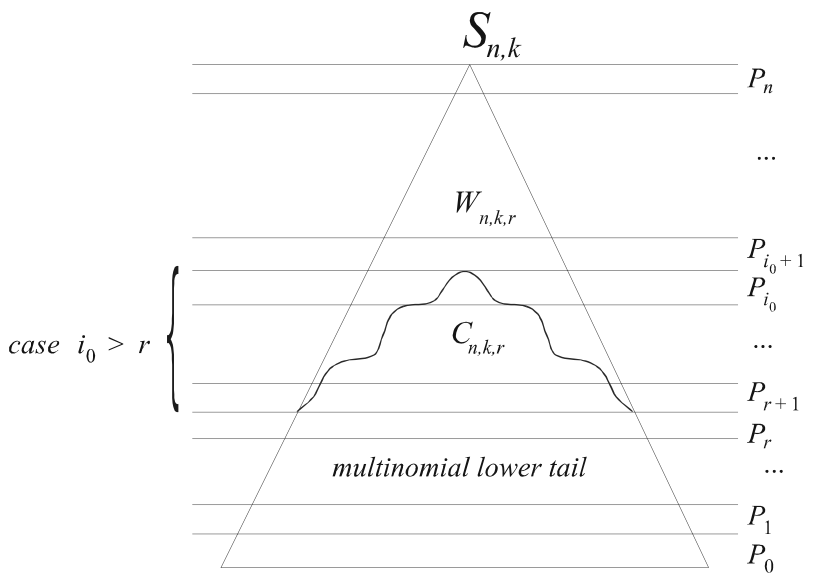

Figure 1
Venn diagram representing the results in Lemma 1 and Lemma 2, for $k \geq 3$

Being uninteresting (and trivial) the case $k = 2$, the two above Lemmas will be proven only for $k \geq 3$.

Proof of Lemma 1.
Assume $(n_1,...,n_k) \in \bigcup_{h=0}^{r} P_h$, $0 \leq r < n$. Then, $n_1 \leq r$, which implies that the condition $n_1 \leq \min_{j=2,...,k} n_j + r$ holds *a fortiori* and, hence, $(n_1,...,n_k) \in C_{n,k,r}$. Besides, for any $h \geq r+1$, the intersection $P_h \cap W_{n,k,r} \neq \varnothing$ is nonempty, since for each $(n_1,...,n_k) \in \bigcup_{h=r+1}^{n} P_h$ where $\min_{j=2,...,k} n_j = 0$, we have $n_1 > \min_{j=2,...,k} n_j + r$. Thus $(P_0, ..., P_r)$, is the "longest" sequence of the type $(P_0, ..., P_j)$, $j \in \{0,...,n-1\}$, included in $C_{n,k,r}$, while P_{r+1}, has at least one element in common with $W_{n,k,r}$.

Proof of Lemma 2.

A1. Consider the case $n < k$**.**
Since for any $(n_1,...,n_k) \in S_{n,k}$, the n_i's sum to n, the condition $(n < k)$ implies that at least one of the $n_1,...,n_k$ is equal to zero: $(n < k) \Rightarrow \min_{j=1,...,k} n = 0$. Therefore, for any $(n_1,...,n_k) \in P_h$, $h = 1,...,n$, we have $n_1 - \min_{j=2,...,k} n_j = n_1$. This implies $P_0 \cup (P_1 \cup ... \cup P_r) \subset C_{n,k,r}$ and $(P_{r+1} \cup ... \cup P_n) \subset W_{n,k,r}$. Thus $i_0 = r$.

A2. Consider the case $(r > n-k)$**.**
When $n_1 > r > n-k$, since it is $\sum_{i=1}^{k} n_i = n$, for each $(n_1, ..., n_k) \in S_{n,k}$, the condition $n_1 > n-k$ implies that at least one of the $n_2, ..., n_k$ is equal to 0. Hence, $n_1 - \min_{j=2,...,k} n_j = n_1 > r$ holds.
This can also be expressed by saying that, for any $r \in \{n-k+1, ..., n-1\}$, the sequence of parts $(P_r, ..., P_n)$ is included in $W_{n,k,r}$, since in each of these parts the condition $n_1 > \min_{j=2,...,k} n_j + r \; (> n-k)$ holds. Thus $i_0 = r$.

Note that, since, by Lemma 1, the union $(P_0 \cup ... \cup P_r)$, $0 \leq r < n$, is included in $C_{n,k,r}$, by definition of i_0 we have: $i_0 \geq r$. On the other hand, since $(P_{n-k+1}, ..., P_n)$ is included in $W_{n,k,r}$, we have: $i_0 \leq n-k$. Therefore:

$$r \leq i_0 \leq n-k \qquad [2.14]$$

B. Consider the case $(n \geq k) \cap (0 \leq r \leq n-k)$**.**
By the definition of i_0, we can write:

$$(i_0 + 1) = \min_{h \in \{1,...,n\}} \left\{ h \mid h > \min_{j=2,...,k} n_j + r \right\}$$

$$= \min_{h \in \{1,\ldots,n\}} \left\{ h \;\mid\; h > [\frac{n-h}{k-1}] + r \right\} \qquad [2.15]$$

by denoting $R_{u,w} \equiv u - w\,[\frac{u}{w}]$ the remainder of the division of *u* by $w,\ w \neq 0$,

$$= \min_{h \in \{1,\ldots,n\}} \left\{ h \;\mid\; h > \frac{n-h-R_{n-h,k-1}}{k-1} + r \right\}. \qquad [2.16]$$

From [2.15], since $R_{n-h,k-1} \leq k-2$, we have:

$$(i_0 + 1) \geq \min_{h \in \{1,\ldots,n\}} \left\{ h \mid h > \frac{n-h-(k-2)}{k-1} + r \right\}$$

$$= \min_{h \in \{1,\ldots,n\}} \left\{ h \mid h > \frac{n+r(k-1)-(k-2)}{k} \right\}$$

$$= [\frac{n+r(k-1)-(k-2)}{k}] + 1 \equiv \alpha_L + 1. \qquad [2.17]$$

On the other hand, from [2.15], since $R_{n-h,k-1} \geq 0$, we have:

$$(i_0 + 1) \leq \min_{h \in \{1,\ldots,n\}} \left\{ h \mid h > \frac{n-h}{k-1} + r \right\}$$

$$= \min_{h \in \{1,\ldots,n\}} \left\{ h \mid h > \frac{n+r(k-1)}{k} \right\}$$

$$= [\frac{n+r(k-1)}{k}] + 1 \equiv \alpha_U + 1. \qquad [2.18]$$

Therefore

$$\alpha_L \leq i_0 \leq \alpha_U . \qquad [2.19]$$

By observing that $\alpha_U - \alpha_L < 1$, since $i_0 \geq r$, we conclude that, depending on *n, k, r*, either one of the following must hold:

$$i_0 = \alpha_U \quad \text{or} \quad i_0 = \alpha_U - 1 \qquad [2.20]$$

In the next Section, we will propose a proof of the Gupta-Nagel conjecture. Since we make use of the partitions [2.2] and [2.5] of the set $S_{n,k}$, we provide some additional observations on them.

Remark 1.2
By combining the results in Lemmas 1 and 2, we have that for $k \geq 3$, the set $S_{n,k}$ can be represented (cf. Figure 1) as a union of $n+1$ disjoint parts (cf. Figure 1):

$$P_0 \cup \ldots \cup P_r \cup P_{r+1} \cup \ldots \cup P_{i_0} \cup P_{i_0+1} \cup \ldots \cup P_n \qquad [2.21]$$

where the two outer sequences $(P_0 \cup \ldots \cup P_r) \subset C_{n,k,r}$, $(P_{i_0+1} \cup \ldots \cup P_n) \subset W_{n,k,r}$ *always exist* and are the *longest* sequences of P_h's contained in $C_{n,k,r}$ and $W_{n,k,r}$, respectively. The inner sequence

$(P_{r+1} \cup ... \cup P_{i_0})$ exists *if and only if* $i_0 > r$. When it exists, we have that, for each P_h in that sequence $(h = r+1,...,i_0)$, the two intersections: $(P_h \cap C_{n,k,r})$ and $(P_h \cap W_{n,k,r})$ are *nonempty* .

The part $P_h \subset S_{n,k}$, where n_1 is equal to *h*, has cardinality, say $| P_h |$, equal to the number of the ordered *partitions* of $(n-h)$ into exactly $(k-1)$ nonnegative integers. Clearly, the cardinality $| P_h |$ is a decreasing function of *h*.

The sets P_0, P_r are, respectively, the largest and the smallest parts of $S_{n,k}$ which are included in $C_{n,k,r}$, while P_{i_0+1}, P_n are, respectively, the largest and the smallest parts of $S_{n,k}$ which are included in $W_{n,k,r}$.

When $i_0 > r$, the sets $(P_{r+1} \cap C_{n,k,r})$ and $(P_{i_0} \cap C_{n,k,r})$ are, respectively, the largest and the smallest intersections of $C_{n,k,r}$ with any of the P_h's, $h = r+1, ..., i_0$.

3. MAIN RESULTS

As a first consequence of Lemmas 1 and 2, we are able to establish easily a novel inequality, which bounds tightly the *multinomial* tail [1.1] and, hence the corresponding linear combination of Dirichlet integrals obtainable by [1.15], between two simple *binomial* tails. The following inequality is of general interest and it also allows to anticipate that, *under some specific conditions*, the Gupta-Nagel conjecture holds true, as shown in Corollary 1.

Theorem 1.

Given $n, k, r, \ n > 0, \ k > 2, \ 0 \le r < n$, let i_0, be the integer defined in Lemma 2.

We have:

$$\frac{\int_0^{1-\theta_1} t^{n-r-1}(1-t)^r dt}{B(n-r, r+1)} \le \sum_{\substack{n_1 \ge 0,...,n_k \ge 0 \\ \sum_{j=1}^k n_j = n \\ n_1 \le \min_{j=2,...,k} n_j + r}} \frac{n!}{n_1! \ ... \ n_k!} \ \theta_1^{n_1} \ ... \ \theta_k^{n_k} \le \frac{\int_0^{1-\theta_1} t^{n-i_0-1}(1-t)^{i_0} dt}{B(n-i_0, i_0+1)} \qquad [3.1]$$

where the equalities hold when $r = i_0$.

Proof of Theorem 1.

Since by definitions [2.1]-[2.4], we have $C_{n,k,r} = S_{n,k} - W_{n,k,r}$, the multinomial lower tail [1.1] can be represented as:

$$\sum_{\substack{n_1 \ge 0,...,n_k \ge 0 \\ \sum_{j=1}^k n_j = n \\ n_1 \le \min_{j=2,...,k} n_j + r}} \frac{n!}{n_1! \ ... \ n_k!} \ \theta_1^{n_1} \ ... \ \theta_k^{n_k}$$

$$= \sum_{C_{n,k,r}} \frac{n!}{n_1! \ ... \ n_k!} \ \theta_1^{n_1} \ ... \ \theta_k^{n_k} \qquad [3.2]$$

or as:

$$= 1 - \sum_{W_{n,k,r}} \frac{n!}{n_1! \ ... \ n_k!} \ \theta_1^{n_1} \ ... \ \theta_k^{n_k} \qquad [3.3]$$

From [3.2], since by Lemma 1, $(P_0, ..., P_r) \subset C_{n,k,r}$ is the "longest" subsequence in $(P_0, ..., P_{n-1})$ which is included in $C_{n,k,r}$, we have:

$$\geq \sum_{P_0 \cup \dots \cup P_r} \frac{n!}{n_1! \dots n_k!} \theta_1^{n_1} \dots \theta_k^{n_k} \qquad [3.4]$$

$$= \sum_{n_1=0}^{r} \binom{n}{n_1} \theta_1^{n_1} (1-\theta_1)^{n-n_1} = \frac{\int_0^{1-\theta_1} t^{n-r-1}(1-t)^r dt}{B(n-r, r+1)} \qquad [3.5]$$

On the other hand, from [3.3], since by Lemma 2, $(P_{i_0+1}, \dots, P_n) \subset W_{n,k,r}$ is the "longest" subsequence in $(P_1, \dots, P_n)$ which is included in $W_{n,k,r}$, we have:

$$\leq 1 - \sum_{P_{i_0+1} \cup \dots \cup P_n} \frac{n!}{n_1! \dots n_k!} \theta_1^{n_1} \dots \theta_k^{n_k} \qquad [3.6]$$

$$= 1 - \sum_{n_1=i_0+1}^{n} \binom{n}{n_1} \theta_1^{n_1} (1-\theta_1)^{n-n_1} \qquad [3.7]$$

$$= \sum_{n_1=0}^{i_0} \binom{n}{n_1} \theta_1^{n_1} (1-\theta_1)^{n-n_1} = \frac{\int_0^{1-\theta_1} t^{n-i_0-1}(1-t)^{i_0} dt}{B(n-i_0, i_0+1)} \qquad [3.8]$$

Finally, note that when $i_0 = r$ the two bounds [3.5] and [3.8] are coincident.
Figure 1 represents clearly the intuitive meaning of the above inequality.

Corollary 1.
When $i_0 = r$ (and, in particular, under the implying condition $(n < k) \cup (r > n-k)$), the Gupta-Nagel conjecture, holds true.

Proof of Corollary 1 (partial proof of Gupta-Nagel conjecture).
When $i_0 = r$, by [3.1], we have that [1.1] is equal to a binomial lower tail, and, hence, is a decreasing function of θ_1, $0 < \theta_1 \leq 1/k$. Besides, by proof of Lemma 2, the condition $(n < k) \cup (r > n-k)$ implies $(i_0 = r)$.

To complete the proof of the conjecture, it remains to show that it holds also under the conditions $i_0 > r$, $k \geq 3$. To this purpose, we present some further results (Lemmas 3 and 4), while Theorem 2 shall complete the argument.

Lemma 3.
For $k \geq 3$, the multinomial lower tail [1.1] is minimized at a point which has necessarily the following form:

$$(\theta_1, \frac{1-\theta_1}{k-1}, \dots, \frac{1-\theta_1}{k-1}) \qquad [3.9]$$

referred to, in the literature, as the *slippage configuration*.

Proof of Lemma 3.
The result in Lemma 3 is due to Gupta and Nagel (1967) (cf. also, Proschan and Sethuraman (1977) and Marshall and Olkin (1979)). For self-containedness, we provide the reader with a concise argument. We have:

$$\tau_r(\theta_1,...,\theta_k) \;\equiv\; \sum_{\substack{n_1 \geq 0,...,n_k \geq 0 \\ \sum_{j=1}^{k} n_j = n \\ n_1 \leq \min_{j=2,...,k} n_j + r}} \frac{n!}{n_1! \;...\; n_k!}\; \theta_1^{n_1} \;...\; \theta_k^{n_k}$$

$$= \sum_{\substack{n_1,..., n_{k-2} \\ \bigcap_{j=2}^{k-2} n_j \,\geq\, n_1 - r}} \frac{n!}{n_1! \,...\, n_{k-2}! \;(n - \sum_{j=1}^{k-2} n_j)!}\; \theta_1^{n_1}...\theta_{k-2}^{n_{k-2}} \sum_{n_k = l_{k-2,r}}^{n-\sum_{j=1}^{k-2} n_j} \begin{pmatrix} n - \sum_{j=1}^{k-2} n_j \\ n_k \end{pmatrix} \left(\frac{\theta_{k-1}}{(1 - \sum_{j=1}^{k-2} \theta_j)} \right)^{n-\sum_{j=1}^{k-2} n_j - n_k} \left(\frac{\theta_k}{(1 - \sum_{j=1}^{k-2} \theta_j)} \right)^{n_k}$$

$$= \sum_{\substack{n_1,..., n_{k-2} \\ \bigcap_{j=2}^{k-2} n_j \geq n_1 - r}} \frac{n!}{n_1! \,...\, n_{k-2}! \;(n - \sum_{j=1}^{k-2} n_j)!}\; \theta_1^{n_1} \,...\, \theta_{k-2}^{n_{k-2}} \;\; \frac{\int_0^{\theta_k/(1-\sum_{j=1}^{k-2}\theta_j)} t^{l_{k,r}-1}\; (1-t)^{n-\sum_{j=1}^{k-2} n_j - l_{k-2,r}}\, dt}{B(l_{k-2,r},\, n - \sum_{j=1}^{k-2} n_j - l_{k-2,r} + 1)} \qquad [3.10]$$

where we have denoted by $l_{k-2,r}$ the smallest value taken by n_k conditionally on $n_1 \,...\, n_{k-2}, n, k, r$.
Thus, for *fixed* $\theta_1 \;...\; \theta_{k-2}$, the multinomial lower tail $\tau_r(\theta_1,...,\theta_k)$ is minimized when θ_k is minimum, i.e., for $\theta_k = \theta_{k-1}$. If $k > 3$, by setting $\theta_k = \theta_{k-1}$ and iterating for $j = 1,...,(k-3)$ times the above argument on the new renormalized arguments, we complete the proof.

Remark 3.1 (*Decomposition into binomial components*)
The reason why we have introduced the partition [2.5] is that, as we shall see, under the slippage configuration, it allows us to express the multinomial probability as a combination of binomial probabilities. In fact, by Lemmas 1 and 2, if we compute the multinomial tail at the slippage configuration, we can write the following "*decomposition into binomial components*":

$$\tau_r\;(\theta_1,\; \frac{1-\theta_1}{k-1},\; ...\;,\; \frac{1-\theta_1}{k-1})$$

$$= \sum_{C_{n,k,r}} \frac{n!}{n_1! \;...\; n_k!}\; \theta_1^{n_1} \left(\frac{1-\theta_1}{k-1}\right)^{n-n_1} \qquad [3.11]$$

$$= \sum_{h=0}^{i_0} \sum_{P_h \cap C_{n,k,r}} \frac{n!}{h! \;...\; n_k!}\; \theta_1^{h} \left(\frac{1-\theta_1}{k-1}\right)^{n-h} \qquad [3.12]$$

$$= \sum_{h=0}^{i_0} \left[\frac{1}{(k-1)^{n-h}} \sum_{P_h \cap C_{n,k,r}} \frac{(n-h)!}{n_2! \;...\; n_k!} \right] \begin{pmatrix} n \\ h \end{pmatrix} \theta_1^h \;(1-\theta_1)^{n-h} \qquad [3.13]$$

this is clearly a weighted sum of binomial probabilities, where the weight for the binomial term $\begin{pmatrix} n \\ h \end{pmatrix} \theta_1^h \;(1-\theta_1)^{n-h}$ is:

$$q_{n,k,r}(h) \;\equiv\; \frac{1}{(k-1)^{n-h}} \sum_{P_h \cap C_{n,k,r}} \frac{(n-h)!}{n_2! \;...\; n_k!}\,,$$

$$= \frac{1}{(k-1)^{n-h}} \sum_{\substack{n_2,...,n_k \\ \sum_{j=1}^{k} n_j = n,\, n_1 = h \\ \min_{j=2,...,k} n_j \geq h - r}} \frac{(n-h)!}{n_2! \;...\; n_k!} \qquad [3.14]$$

which we will refer to as a "weight of order h in the decomposition into binomial components", and below we illustrate a mathematical interpretation of $q_{n,k,r}(h)$.

Also, we denote the ordinary binomial term of order h by:

$$BinT_h(\theta_1) \equiv \binom{n}{h} \theta_1^h \ (1-\theta_1)^{n-h}$$
$$= \sum_{P_h} \frac{n!}{n_1! \ \dots \ n_k!} \ \theta_1^{n_1} \left(\frac{1-\theta_1}{k-1}\right)^{n-n_1} \qquad [3.15]$$

Using the above notation, we can finally write our decomposition [3.13] concisely as:

$$\tau_r(\theta_1,\dots,\theta_k) = \sum_{h=0}^{i_0} q_{n,k,r}(h) \ BinT_h(\theta_1) \qquad [3.16]$$

Remark 3.2 (*Behavior of the binomial probabilities*)
Note that the binomial terms $BinT_h(\theta)$, $h = 1,\dots,n-1$, $\theta \in (\ 0, \frac{1}{k}]$ are increasing for $\theta < \frac{h}{n}$ and decreasing for $\theta > \frac{h}{n}$, while $BinT_0(\theta)$, $BinT_n(\theta)$ are, respectively, decreasing and increasing (cf. Figure 2, in Section 7).

Notation. ***Relative sum of "central" multinomial coefficients***

The weights $q_{n,k,r}(h)$ by definition [3.14] consider a bound $h-r$ for the minimum frequency $\min_{j=2,\dots,k} n_j$ which depends on h . It might be useful to define a "generalized" form of the weight $q_{n,k,r}(h)$ where h and the bound for the minimum frequency are not necessarily linked, and therefore the influence of each argument can be more easily studied. Therefore, we also introduce the symbol:

$$Genq_{sum,k,bound}(h) \equiv \frac{1}{(k-1)^{sum-h}} \sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = sum, \ n_1 = h \\ \min_{j=2,\dots,k} n_j \geq bound}} \frac{(sum-h)!}{n_2! \ \dots \ n_k!} \qquad [3.17]$$

which we might refer to as a "generalized weight of order h", and, clearly, the following "wrapping" relation holds:

$$q_{n,k,r}(h) \equiv Genq_{sum,k,bound}(h) \text{ , where } sum = n, \ bound = h - r \, . \qquad [3.18]$$

Besides its statistical meaning as weight, the function $Genq_{sum,k,bound}(h)$ has also an interesting mathematical interpretation. In fact, for instance when $k = 3$, the weight $Genq_{sum,3,bound}(h)$ is the sum of some (depending on $bound$) "central" coefficients on the row $sum - h$ of the Pascal triangle divided the sum of the coefficients of the whole row, i.e., it represents the "relative" sum of some "central" coefficients in the expansion of $(x+y)^{sum-h}$. Similarly, $Genq_{sum,k,bound}(h)$ can be seen as the analogous relative sum of "central" multinomial coefficients in the expansion of $(x_1 + \dots + x_{k-1})^{sum-h}$ (in this case, we might visualize the extension of the Pascal triangle as a tetrahedron ($k = 4$), a pentachoron ($k = 5$), or, in general, a ($k-1$)-simplex). Therefore, the properties of these sum of "central" multinomial coefficients, apart the specific problem we are dealing with, can be also of general mathematical interest.

The following Lemma 4 is concerned with the properties of the above weights. These properties will be used to provide a monotonicity argument for the multinomial tail under the slippage configuration and they can be useful to solve other similar problems, as shown in section 4.

Lemma 4.

Given $n, k, r, \; n > 0, \; k \geq 3, \; 0 \leq r < n$, let i_0, $r \leq i_0 < n$, be the integer defined in Lemma 2, we have:

A) $$q_{n,k,r}(h) = 1, \text{ for } h = 0, \dots, r, \qquad [3.19]$$

B) $$0 < q_{n,k,r}(h) < 1, \text{ for } h = r+1, \dots, i_0, r < i_0, \qquad [3.20]$$

C) $$q_{n,k,r}(h) = 0, \text{ for } h = i_0 + 1, \dots, n, \qquad [3.21]$$

D) ***Recurrence relation***. The following interesting recurrence equation, which links the weight of order h with $k+1$ cells with the weight of same order h when the number of cells is k, holds:

$$q_{n,k+1,r}(h) = \begin{cases} = (\frac{k-1}{k})^{n-h} \sum_{u=max(0,h-r)}^{n-h} \binom{n-h}{u} (\frac{1}{k-1})^{u} \; q_{n-u,k,r}(h) & if \quad 0 \leq h \leq [\frac{n+r}{2}] \\ = 0 \quad if \quad h > [\frac{n+r}{2}] \end{cases}$$

[3.22]

E) The weights $q_{n,k,r}(h)$, $h = 0, \dots, n$ form a non increasing sequence from 1 to 0, and in particular, for the central values $h = r+1, \dots, i_0$, $r < i_0$, $q_{n,k,r}(h)$ is a *decreasing* function of h.

F) ***Order transition formula***

$$Genq_{sum+1,k,bound}(h+1) = Genq_{sum,k,bound}(h) \qquad [3.23]$$

G) ***Increase by bound release***

$$Genq_{sum,k,bound}(h) < Genq_{sum,k,bound-1}(h) \qquad [3.24]$$

For an illustration of all the above relationships, cf. Section 7, Tables 3-6.

Remarks and notation preliminary to the proof of Lemma 4

Remark 3.3

We can write the weight $q_{n,k,r}(h)$ as follows:

$$q_{n,k,r}(h) \equiv \frac{1}{(k-1)^{n-h}} \sum_{P_h \cap C_{n,k,r}} \frac{(n-h)!}{n_2! \; \dots \; n_k!} \qquad [3.25]$$

$$= \frac{1}{(k-1)^{n-h}} \sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = n,\, n_1 = h \\ \min_{j=2,\dots,k} n_j \geq h-r}} \frac{(n-h)!}{n_2! \ \dots \ n_k!} \qquad [3.26]$$

denoted as:

$$\equiv \frac{1}{(k-1)^{n-h}} \ SumC_{n,k,r}(h) \qquad [3.27]$$

or equivalently, by observing that $\frac{1}{(k-1)^{n-h}} \sum_{P_h} \frac{(n-h)!}{n_2! \ \dots \ n_k!} = 1$, [3.28]

$$q_{n,k,r}(h) = 1 - \frac{1}{(k-1)^{n-h}} \sum_{P_h \cap W_{n,k,r}} \frac{(n-h)!}{n_2! \ \dots \ n_k!} \qquad [3.29]$$

$$= 1 - \frac{1}{(k-1)^{n-h}} \sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = n,\, n_1 = h \\ \min_{j=2,\dots,k} n_j < h-r}} \frac{(n-h)!}{n_2! \ \dots \ n_k!} \qquad [3.30]$$

denoted as:

$$\equiv 1 - \frac{1}{(k-1)^{n-h}} \ SumW_{n,k,r}(h) \qquad [3.31]$$

Proof of Lemma 4.

A.
Consider expression [3.31]. For $h = 0, \dots, r$, $q_{n,k,r}(h) = 1$ because, by Lemma 1, $P_h \cap W_{n,k,r} = \varnothing$.

B.
Assume $h = r+1, \dots, i_0$, $r < i_0$. In such a case, by Lemmas 1 and 2, both sets $(P_h \cap C_{n,k,r})$ and $(P_h \cap W_{n,k,r})$ are nonempty, hence by [3.31], [3.27], we have $0 < q_{n,k,r}(h) < 1$.

C.
Consider expression [3.27]. For $h = i_0 + 1, \dots, n$, $q_{n,k,r}(h) = 0$, because, by Lemma 2, $P_h \cap C_{n,k,r} = \varnothing$

D. (*Recurrence relation*)
The following proof is also illustrated Section 7, Tables 1-2.
By [3.29]-[3.31] we have:

$$k^{n-h} \ q_{n,k+1,r}(h) = k^{n-h} - SumW_{n,k+1,r}(h) \qquad [3.32]$$

$$= k^{n-h} - \sum_{\substack{n_2,\dots,n_{k+1} \\ \sum_{j=1}^{k+1} n_j = n,\, n_1 = h \\ \min_{j=2,\dots,k+1} n_j < h-r}} \frac{(n-h)!}{n_2! \ \dots \ n_{k+1}!} \qquad [3.33]$$

$$= \quad k^{n-h} \quad - \sum_{\substack{n_2,\dots,n_{k+1} \\ \sum_{j=1}^{k+1} n_j = n, n_1 = h \\ \min_{j=2,\dots,k} n_j < h-r \;\cup\; n_{k+1} < h-r}} \frac{(n-h)!}{n_2! \;\dots\; n_{k+1}!} \tag{3.34}$$

$$= \quad k^{n-h} \quad - \sum_{\substack{n_2,\dots,n_{k+1} \\ \sum_{j=1}^{k+1} n_j = n, n_1 = h \\ \min_{j=2,\dots,k} n_j < h-r \;\bigcup\; (n_{k+1} < h-r \;\cap\; (\min_{j=2,\dots,k} n_j < h-r)^c)}} \frac{(n-h)!}{n_2! \;\dots\; n_{k+1}!} \tag{3.35}$$

$$= \quad k^{n-h} \quad - \Bigg(\sum_{\substack{n_2,\dots,n_{k+1} \\ \sum_{j=1}^{k+1} n_j = n, n_1 = h \\ \min_{j=2,\dots,k} n_j < h-r}} \frac{(n-h)!}{n_2! \;\dots\; n_{k+1}!} + \sum_{\substack{n_2,\dots,n_{k+1} \\ \sum_{j=1}^{k+1} n_j = n, n_1 = h \\ (n_{k+1} < h-r \;\cap\; \min_{j=2,\dots,k} n_j \ge h-r)}} \frac{(n-h)!}{n_2! \;\dots\; n_{k+1}!} \Bigg) \tag{3.36}$$

which will be denoted as follows:

$$= \quad k^{n-h} \quad - \quad (SumW1_{n,k+1,r} + SumW2_{n,k+1,r}) \tag{3.37}$$

We note that, for $h > r$, the following relationship holds:

$$SumW1_{n,k+1,r} \equiv \sum_{\substack{n_2,\dots,n_{k+1} \\ \sum_{j=1}^{k+1} n_j = n, n_1 = h \\ \min_{j=2,\dots,k} n_j < h-r}} \frac{(n-h)!}{n_2! \;\dots\; n_{k+1}!} \tag{3.38}$$

$$= \sum_{u=0}^{n-h} \binom{n-h}{u} \sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = n-u, n_1 = h \\ \min_{j=2,\dots,k} n_j < (h+u)-r}} \frac{((n-u)-h)!}{n_2! \dots \; n_k!} \tag{3.39}$$

$$= \sum_{u=0}^{n-h} \binom{n-h}{u} \; SumW_{n-u,k,r} \tag{3.40}$$

$$= \sum_{u=0}^{n-h} \binom{n-h}{u} \; (k-1)^{n-u-h} \; (1 - q_{n-u,k,r}(h)) \; . \tag{3.41}$$

On the other hand, we have:

$$SumW2_{n,k+1,r} \equiv \sum_{\substack{n_2,\dots,n_{k+1} \\ \sum_{j=1}^{k+1} n_j = n, n_1 = h \\ (n_{k+1} < h-r \;\cap\; \min_{j=2,\dots,k} n_j \ge h-r)}} \frac{(n-h)!}{n_2! \;\dots\; n_{k+1}!} \tag{3.42}$$

$$= \sum_{u=0}^{h-r-1} \sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = n-u,\ n_1 = h \\ (u<h-r \ \cap \ \min_{j=2,\dots,k} n_j \geq h-r)}} \frac{(n-h)!}{n_2! \ \dots \ n_k! \ u!} \qquad [3.43]$$

$$= \sum_{u=0}^{h-r-1} \binom{n-h}{u} \sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = n-u,\ n_1 = h \\ (u<h-r \ \cap \ \min_{j=2,\dots,k} n_j \geq h-r)}} \frac{((n-u)-h)!}{n_2! \dots \ n_k!} \qquad [3.44]$$

$$= \sum_{u=0}^{h-r-1} \binom{n-h}{u} \sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = n-u,\ n_1 = h \\ \min_{j=2,\dots,k} n_j \geq h-r}} \frac{((n-u)-h)!}{n_2! \dots \ n_k!} \qquad [3.45]$$

$$= \sum_{u=0}^{h-r-1} \binom{n-h}{u} (k-1)^{(n-u)-h} \ q_{n-u,k,r}(h) \ . \qquad [3.46]$$

By substituting [3.41] and [3.46] into [3.37], we obtain:

$$\begin{aligned} k^{n-h} \ q_{n,k+1,r}(h) = \ & k^{n-h} \ - \ (\sum_{u=0}^{n-h} \binom{n-h}{u} (k-1)^{n-u-h} \ (1 - q_{n-u,k,r}(h)) \\ & + \sum_{\substack{u=0 \\ u \leq n-h}}^{h-r-1} \binom{n-h}{u} (k-1)^{n-u-h} \ q_{n-u,k,r}(h) \ ) \end{aligned} \qquad [3.47]$$

$$\begin{aligned} = \ & k^{n-h} \ - \ (k-1)^{n-h} \sum_{u=0}^{n-h} \binom{n-h}{u} (\frac{1}{k-1})^{u} \\ & + \sum_{u=0}^{n-h} \binom{n-h}{u} (k-1)^{n-u-h} \ q_{n-u,k,r}(h) \\ & - \sum_{\substack{u=0 \\ u \leq n-h}}^{h-r-1} \binom{n-h}{u} (k-1)^{n-u-h} \ q_{n-u,k,r}(h) \end{aligned} \qquad [3.48]$$

$$= \sum_{u=0}^{n-h} \binom{n-h}{u} (k-1)^{n-u-h} \ q_{n-u,k,r}(h) \ - \sum_{\substack{u=0 \\ u \leq n-h}}^{h-r-1} \binom{n-h}{u} (k-1)^{n-u-h} \ q_{n-u,k,r}(h) \qquad [3.49]$$

if $h-r-1 < n-h$ (i.e., $h \leq [\frac{n+r}{2}]$)

$$= \sum_{u=\max(0,h-r)}^{n-h} \binom{n-h}{u} (k-1)^{n-u-h} \ q_{n-u,k,r}(h) \qquad [3.50]$$

Therefore, we can conclude:

$$k^{n-h}\ q_{n,k+1,r}(h) = \begin{cases} = \sum_{u=\max(0,h-r)}^{n-h} \binom{n-h}{u} (k-1)^{n-u-h}\ q_{n-u,k,r}(h), & if\ \ 0 \le h \le [\frac{n+r}{2}] \\ = 0,\ if\ \ h > [\frac{n+r}{2}] \end{cases}$$

[3.51]

E. ***Order transition formula***

$$Genq_{sum,k,bound}(h) \equiv \frac{1}{(k-1)^{sum-h}} \sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = sum,\ n_1 = h \\ \min_{j=2,\dots,k} n_j \ge bound}} \frac{(sum-h)!}{n_2!\ \dots\ n_k!} \quad [3.52]$$

If we consider the two sets:

$$\left\{(h, n_2, \dots, n_k) \mid \sum_{j=1}^{k} n_j = sum,\ \min_{j=2,\dots,k} n_j \ge bound\right\} \quad [3.53]$$

$$\left\{(h+1, n_2, \dots, n_k) \mid \sum_{j=1}^{k} n_j = sum+1,\ \min_{j=2,\dots,k} n_j \ge bound\right\} \quad [3.54]$$

a one-to-one mapping can by established $(h, n_2, \dots, n_k) \Leftrightarrow (h+1, n_2, \dots, n_k)$ and [3.52] can be written as:

$$= \frac{1}{(k-1)^{sum-h}} \sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = sum+1,\ n_1 = h+1 \\ \min_{j=2,\dots,k} n_j \ge bound}} \frac{(sum-h)!}{n_2!\ \dots\ n_k!} \quad [3.55]$$

$$= \frac{1}{(k-1)^{(sum+1)-(h+1)}} \sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = sum+1,\ n_1 = h+1 \\ \min_{j=2,\dots,k} n_j \ge bound}} \frac{((sum+1)-(h+1))!}{n_2!\ \dots\ n_k!} \quad [3.56]$$

$$= Genq_{sum+1,k,bound}(h+1) \quad [3.57]$$

F. ***Increase by bound release***

$$Genq_{n,k,sum,bound}(h)$$

$$\equiv \frac{1}{(k-1)^{n-h}} \sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = sum,\ n_1 = h \\ \min_{j=2,\dots,k} n_j \ge bound}} \frac{(n-h)!}{n_2!\ \dots\ n_k!} \quad [3.58]$$

since $(\min_{j=2,\dots,k} n_j \ge bound) \Rightarrow (\min_{j=2,\dots,k} n_j \ge bound - 1)$, i.e. the condition $\min_{j=2,\dots,k} n_j \ge bound$ is more stringent than $\min_{j=2,\dots,k} n_j \ge bound - 1$, we have:

$$\left\{(h,n_2,\dots,n_k) \mid \sum_{j=2}^{k} n_j = sum - h,\ \min_{j=2,\dots,k} n_j \geq bound \right\}$$
$$\subset \left\{(h,n_2,\dots,n_k) \mid \sum_{j=2}^{k} n_j = sum - h,\ \min_{j=2,\dots,k} n_j \geq bound - 1 \right\}$$

and therefore:

$$> \frac{1}{(k-1)^{n-h}} \sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = sum,\ n_1 = h \\ \min_{j=2,\dots,k} n_j \geq bound-1}} \frac{(n-h)!}{n_2!\ \dots\ n_k!} \qquad [3.59]$$

$$\equiv\ Genq_{n,k,sum,bound-1}(h)$$

G. *Weight Sequence monotonicity*

In points A) B) C) we have seen that the weights $q_{n,k,r}(h)$ are always in [0,1], and, in particular, they are equal to 1 for $h = 0, \dots, r$ and equal to 0 for $h = i_0 + 1, \dots, n$. We show now that, for any intermediate value $h = r+1, \dots, i_0$, $r < i_0$, the sequence of weights $q_{n,k,r}(h)$ is decreasing.

We have:

$$q_{n,k,r}(h) \ -\ q_{n,k,r}(h+1) \qquad [3.60]$$

$$= \frac{1}{(k-1)^{n-h}} \sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = n, n_1 = h \\ \min_{j=2,\dots,k} n_j \geq h-r}} \frac{(n-h)!}{n_2!\ \dots\ n_k!} - \frac{1}{(k-1)^{n-(h+1)}} \sum_{\substack{\nu_2,\dots,\nu_k \\ \sum_{j=1}^{k} \nu_j = n, \nu_1 = h+1 \\ \min_{j=2,\dots,k} \nu_j \geq (h+1)-r}} \frac{(n-(h+1))!}{\nu_2!\ \dots\ \nu_k!} \qquad [3.61]$$

$$= \frac{1}{(k-1)^{n-h}} \sum_{P_h \cap C_{n,k,r}} \frac{(n-h)!}{n_2!\ \dots\ n_k!} - \frac{1}{(k-1)^{n-(h+1)}} \sum_{P_{h+1} \cap C_{n,k,r}} \frac{(n-(h+1))!}{\nu_2!\ \dots\ \nu_k!} \qquad [3.62]$$

For any h and any integer $release \geq 0$, let us denote by

$$R_{n,k,r,h,release} \equiv \left\{ (h,n_2,\dots,n_k) \in P_h \mid \sum_{j=2}^{k} n_j = n - h,\ \min_{j=2,\dots,k} n_j \geq h - r - release \right\} \qquad [3.63]$$

Note that $R_{n,k,r,h,release}$ is a possibly augmented version of $P_h \cap C_{n,k,r}$ where, in case there is a positive release of the bound $h - r$, the configurations are a larger number:

$$R_{n,k,r,h,release} \supset P_h \cap C_{n,k,r}, \qquad [3.64]$$

and, in particular,

$$R_{n,k,r,h,0} = P_h \cap C_{n,k,r}. \qquad [3.65]$$

With the above notation, we can write [3.62] as:

$$= \frac{1}{(k-1)^{n-h}} \sum_{(h,n_2,\ldots,n_k)\in P_h \cap C_{n,k,r}} \frac{(n-h)!}{n_2! \ \ldots \ n_k!} - \frac{1}{(k-1)^{n-(h+1)}} \sum_{(h+1,\nu_2,\ldots,\nu_k)\in R_{n,k,r,h+1,0}} \frac{(n-(h+1))!}{\nu_2! \ \ldots \ \nu_k!} \quad [3.66]$$

Consider the two sets:

$$P_h \cap C_{n,k,r} = R_{n,k,r,h,0} \equiv \left\{ (h,n_2,\ldots,n_k) \in P_h \mid \sum_{j=2}^{k} n_j = n-h, \ \min_{j=2,\ldots,k} n_j \geq h-r \right\} \quad [3.67]$$

$$R_{n,k,r,h+1,release} \equiv \left\{ (h+1,\nu_2,\ldots,\nu_k) \in P_{h+1} \mid \sum_{j=2}^{k} \nu_j = n-(h+1), \ \min_{j=2,\ldots,k} \nu_j \geq (h+1)-r-release \right\}$$

[3.68]

over which the two sums are carried out. Note that for $h = r+1,\ldots,i_0$, $r < i_0$, by Lemmas 1-2, the set $(P_h \cap C_{n,k,r})$ is not empty. Then consider any $release \geq 0$. Since, to any configuration $(h,n_2,\ldots,n_k) \in P_h \cap C_{n,k,r}$, it might correspond a configuration $(h+1,n_2-1,\ldots,\nu_k) \in R_{n,k,r,h+1,release}$ and vice versa, it is possible to establish a *one-to-one* correspondence between a subset of $(P_h \cap C_{n,k,r})$ and a subset of $R_{n,k,r,h+1,release}$:

$$(h,n_2,n_3,\ldots,n_k) \Leftrightarrow (h+1,\nu_2,\nu_3\ldots,\nu_k) = (h+1,n_2-1, \ n_3,\ldots,n_k) \quad [3.69]$$

With respect to the above correspondence, let us denote by $I_{bw}(R_{n,k,r,h+1,release})$ the image of $R_{n,k,r,h+1,release}$ in $(P_h \cap C_{n,k,r})$ and by $I_{fw}(P_h \cap C_{n,k,r})$ the image of $(P_h \cap C_{n,k,r})$ in $R_{n,k,r,h+1,release}$. Note that, if we increase $release$ enough, it is possible to obtain a coincidence between $I_{bw}(R_{n,k,r,h+1,release})$ and the whole set $(P_h \cap C_{n,k,r})$. In particular, given [3.69], it is necessary to take $release > 1$, while $I_{bw}(R_{n,k,r,h+1,release}) \subset (P_h \cap C_{n,k,r})$ for $release = 0,1$. Then, assume $release > 1$ and let us denote by $SubR_{n,k,r,h+1,release}$ the subset of configurations in $R_{n,k,r,h+1,release}$ which are in a one-to-one correspondence with $(P_h \cap C_{n,k,r})$. We have: $I_{fw}(P_h \cap C_{n,k,r}) = SubR_{n,k,r,h+1,release} \supset R_{n,k,r,h+1,1} \supset R_{n,k,r,h+1,0} \equiv (P_{h+1} \cap C_{n,k,r})$.

Therefore, we can write [3.66] as:

$$> \sum_{(h,n_2,\ldots,n_k)\in P_h \cap C_{n,k,r}} \frac{1}{(k-1)^{n-h}} \frac{(n-h)!}{n_2! \ \ldots \ n_k!} - \sum_{(h+1,\nu_2,\ldots,\nu_k)\in I_{fw}(P_h \cap C_{n,k,r})} \frac{1}{(k-1)^{n-(h+1)}} \frac{(n-(h+1))!}{\nu_2! \ \ldots \ \nu_k!} \quad [3.70]$$

$$= \sum_{P_h \cap C_{n,k,r}} \left(\frac{1}{(k-1)^{n-h}} \frac{(n-h)!}{n_2! \ \ldots \ n_k!} - \frac{1}{(k-1)^{n-(h+1)}} \frac{(n-(h+1))!}{(n_2-1)! \ \ldots \ n_k!} \right) \quad [3.71]$$

$$= \sum_{\substack{n_2,\ldots,n_k \\ \sum_{j=1}^{k} n_j = n, n_1 = h \\ \min_{j=2,\ldots,k} n_j \geq h-r}} \frac{1}{(k-1)^{n-h}} \frac{(n-h)!}{n_2! \ \ldots \ n_k!} - \sum_{\substack{n_2,\ldots,n_k \\ \sum_{j=1}^{k} n_j = n, n_1 = h \\ \min_{j=2,\ldots,k} n_j \geq h-r}} \frac{1}{(k-1)^{n-h-1}} \frac{(n-h-1)!}{(n_2-1)! \ \ldots \ n_k!}$$

[3.72]

$$= \sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = n, n_1 = h \\ \min_{j=2,\dots,k} n_j \geq h-r}} \frac{1}{(k-1)^{n-h}} \frac{(n-h)!}{n_2! \ \dots \ n_k!} - \sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = n, n_1 = h \\ \min_{j=2,\dots,k} n_j \geq h-r}} \frac{(k-1)\ n_u}{n-h} \frac{1}{(k-1)^{n-h}} \frac{(n-h)!}{n_2! \ \dots \ n_k!}$$

by simmetry, for any $u \in \{2,\dots,k\}$. Hence:

$$= \frac{1}{k-1} \sum_{u=2}^{k} \left(\sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = n, n_1 = h \\ \min_{j=2,\dots,k} n_j \geq h-r}} \frac{1}{(k-1)^{n-h}} \frac{(n-h)!}{n_2! \ \dots \ n_k!} - \sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = n, n_1 = h \\ \min_{j=2,\dots,k} n_j \geq h-r}} \frac{(k-1)\ n_u}{n-h} \frac{1}{(k-1)^{n-h}} \frac{(n-h)!}{n_2! \ \dots \ n_k!} \right)$$

$$= \frac{1}{k-1} \sum_{u=2}^{k} \left(\sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = n, n_1 = h \\ \min_{j=2,\dots,k} n_j \geq h-r}} \frac{1}{(k-1)^{n-h}} \frac{(n-h)!}{n_2! \ \dots \ n_k!} \quad (1 - \frac{(k-1)\ n_u}{n-h}) \right) = 0$$

Therefore, for $h = r+1,\dots,i_0 - 1$, we have:

$$q_{n,k,r}(h) \ > \ q_{n,k,r}(h+1) \qquad [3.73]$$

Remark 3.5. One might note that the proof of weight sequence monotonicity discloses a surprising property of the relative sums of "central" multinomial weights. The central coefficients in the expansion $(x_1,\dots,x_{k-1})^{sum-h}$ and the corresponding extended set (which we might call a "forward image") which contains, as subset, the central coefficients in the expansion $(x_1,\dots,x_{k-1})^{sum-(h+1)}$ have equal *relative sums*.

At this point, for analogy, one could ask: if we consider the central coefficients in the expansion $(x_1,\dots,x_{k-1})^{sum-h}$, do the corresponding coefficients in the expansion $(x_1,\dots,x_{k-1})^{sum-(h-1)}$ have also equal relative sums? And, since the answer for the "forward image" is affirmative, one could expect that also the "backward image" would have the same property. However, it can be noted that such a property does not hold (see for instance Table 9, in Section 7, where the coefficients in the expansion $(x_1,\dots,x_{k-1})^{sum-(h-1)}$ corresponding to those in the expansion $(x_1,\dots,x_k)^{sum-h}$ have a smaller *relative sum*).

Finally, in order to complete the proof of the Gupta-Nagel conjecture, it is necessary to show that the multinomial tail [1.1], with $\theta_2 = \dots = \theta_k$, is a strictly decreasing function of θ_1, also under the conditions $0 \leq r < i_0, k \geq 3$.

Theorem 2.
Given the integers $n, k, r, \; n > 0, \; k \geq 3, \; 0 \leq r < i_0$, the multinomial tail under the slippage configuration:

$$\tau_r \; (\theta_1, \; \frac{1-\theta_1}{k-1}, \; \dots \; , \frac{1-\theta_1}{k-1}) \qquad [3.74]$$

is *decreasing* with respect to θ_1, $\theta_1 \in (0, \frac{1}{k}]$.

Proof of Theorem 2.
Assume $n > 0, k \geq 3, 0 \leq r < i_0$.
Let us start from our decomposition into binomial components:

$$\tau_r(\theta_1, \dots, \theta_k) = \sum_{h=0}^{i_0} \quad q_{n,k,r}(h) \;\; BinT_h(\theta_1) \qquad [3.75]$$

A study of the monotonicity of the above weighted sum of binomial probabilities is carried out by partitioning the interval $(0, \frac{1}{k}]$ into disjoint subintervals $\theta_1 \in (\frac{i}{n}, \frac{i+1}{n}]$ of length $\frac{1}{n}$ and by comparing the above sum with the ordinary binomial tail:

$$\sum_{h=0}^{i_0} \; BinT_h(\theta_1) \; . \qquad [3.76]$$

As to the ordinary binomial tail, since it is decreasing, for any $\theta_1 \in (0, \frac{1}{k}]$, we have:

$$\frac{\delta}{\delta\theta_1} \sum_{h=0}^{i_0} \; BinT_h(\theta_1) = \frac{\delta}{\delta\theta_1} \; \frac{\int_0^{1-\theta_1} \quad t^{n-i_0-1} \;\; (1-t)^{i_0} \;\; dt}{B(n-i_0, \; i_0+1)} < 0 \qquad [3.77]$$

Hence, for $r < i_0$, we can write:

$$- \sum_{h=0}^{i} \; \frac{\delta}{\delta\theta_1} \; BinT_h(\theta_1) \quad > \quad \sum_{i+1}^{i_0} \; \frac{\delta}{\delta\theta_1} \; BinT_h(\theta_1) \qquad [3.78]$$

for each $0 \leq i \leq i_0$.

By Remark 3.2, the components $BinT_0(\theta_1), \dots, BinT_i(\theta_1)$ are decreasing in the subinterval $(\frac{i}{n}, \frac{i+1}{n}]$, i.e., $\frac{\delta}{\delta\theta_1} BinT_0(\theta_1) < 0, \dots, \frac{\delta}{\delta\theta_1} BinT_i(\theta_1) < 0$, and by Lemma 4, $0 \; > \; q_{n,k,r}(0) \geq \dots \; \geq q_{n,k,r}(i)$, hence we have:

$$- \; \frac{\delta}{\delta\theta_1} \; \sum_{h=0}^{i} \; BinT_h(\theta_1) \;\; q_{n,k,r}(h)$$

$$= \; \sum_{h=0}^{i} \left(- \;\; q_{n,k,r}(h) \; \frac{\delta}{\delta\theta_1} \; BinT_h(\theta_1) \right) \qquad [3.79]$$

$$\geq \; \sum_{h=0}^{i} \left(- \;\; q_{n,k,r}(i) \; \frac{\delta}{\delta\theta_1} \; BinT_h(\theta_1) \right) \qquad [3.80]$$

$$= \; q_{n,k,r}(i) \; \left(- \sum_{h=0}^{i} \; \frac{\delta}{\delta\theta_1} \; BinT_h(\theta_1) \right) \qquad [3.81]$$

$$\geq \ q_{n,k,r}(i+1) \ \left(- \sum_{h=0}^{i} \ \frac{\delta}{\delta\theta_1} \ BinT_h(\theta_1) \right) \tag{3.82}$$

by [3.78]

$$> \ q_{n,k,r}(i+1) \sum_{h=i+1}^{i_0} \ \frac{\delta}{\delta\theta_1} \ BinT_h(\theta_1) \tag{3.83}$$

Since in $(\frac{i}{n}, \frac{i+1}{n}]$ all the components $BinT_{i+1}(\theta_1),..., BinT_{i_0}(\theta_1)$ are increasing, i.e., $\frac{\delta}{\delta\theta_1} \ BinT_{i+1}(\theta_1) \ > \ 0, ..., \frac{\delta}{\delta\theta_1} \ BinT_{i_0}(\theta_1) \ > \ 0$ and, by Lemma 4, $0 \ \geq \ q_{n,k,r}(i+1) \ \geq \ ... \ \geq \ q_{n,k,r}(i_0)$, we can write:

$$\geq \sum_{h=i+1}^{i_0} \ q_{n,k,r}(h) \ \frac{\delta}{\delta\theta_1} \ BinT_h(\theta_1) \tag{3.84}$$

$$= \frac{\delta}{\delta\theta_1} \sum_{h=r+i}^{i_0} BinT_h(\theta_1) \ \ q_{n,k,r}(h)\,. \tag{3.85}$$

Thus, for $r < i_0$, we have:

$$- \frac{\delta}{\delta\theta_1} \sum_{h=0}^{i-1} BinT_{r+h}(\theta_1) \quad q_{n,k,r}(r+h) \quad > \quad \frac{\delta}{\delta\theta_1} \sum_{h=i}^{i_0} \ BinT_{r+h}(\theta_1) \quad q_{n,k,r}(r+h)\,. \tag{3.86}$$

Therefore, in each subinterval $\theta_1 \in (\frac{i}{n}, \frac{i+1}{n}]$, we have:

$$\frac{\delta}{\delta\theta_1} \ \ \tau_r \left(\theta_1, \frac{1-\theta_1}{k-1}, \ ... \ , \frac{1-\theta_1}{k-1}\right)$$

$$= \frac{\delta}{\delta\theta_1}\left(\sum_{h=0}^{i-1} \ BinT_{r+h}(\theta_1) \ \ q_{n,k,r}(r+h) \ \ + \ \sum_{h=i}^{i_0-r} \ BinT_{r+h}(\theta_1) \ \ q_{n,k,r}(r+h) \right) \tag{3.87}$$

$$= \ \frac{\delta}{\delta\theta_1} \ \left(\frac{\int_0^{1-\theta_1} \ t^{n-r-1} \ (1-t)^r \ dt}{B(n-r, \ r+1)} \ + \ \sum_{h=r+1}^{i_0} \ BinT_h(\theta_1) \ \ q_{n,k,r}(h) \right) \ < \ 0 \tag{3.88}$$

Having proven that, in each subinterval of $\theta_1 \in (0, \frac{1}{k}]$, the continuous function [3.30] is decreasing, Corollary 1, Lemma 3 , and Theorem 2 complete the proof of the Gupta-Nagel conjecture.

4. DISCUSSION

We may note that, in the proof of Theorem 2, we have used the fact that the sequence of the weights $q_{n,k,r}(h)'s$ in the decomposition into binomial components is nonincreasing, and *not* the actual values of the $q_{n,k,r}(h)'s$. This means that the method we have employed here for the Gupta-Nagel conjecture is general, and it could be applied to minimize several "variations" of the multinomial tail [1.1], even with "irregular shapes". It is sufficient that Lemma 3 holds (the slippage configuration is quite common in multinomial problems) and that the *relative* sums of multinomial coefficients contained in the intersections $(P_h \cap C_{n,k,r})$, say $q^*_{n,k,r}(h)$, $h = 0,...,n$ (i.e., the *weights* assigned to the "binomial components" forming the tail), are a *nonincreasing* sequence, from $q^*_{n,k,r}(0) = 1$ to $q^*_{n,k,r}(n) = 0$ (in other words, the tail includes the leading decreasing term $(1-\theta)^n$ and excludes the increasing term θ^n).

With the help of Figure 1, we may observe that a sequence of sets $(P_h \cap C_{n,k,r})$, $h = 0,...,n$, which gives rise to monotonic weights $q^*_{n,k,r}(h)'s$, can be chosen in $S_{n,k}$ in a number of ways and "shapes" $\bigcup_{h=0}^{n} (P_h \cap C_{n,k,r})$ in $S_{n,k}$. The class of all possible the above multinomial sums describes an entire family of "multinomial tails", to which the argument developed here can be extended.

Example of minimization of another multinomial tail

Denote:

$$q'_{n,k,r}(h,\delta) \equiv 1, \text{ for } h = 0,...,r \qquad [4.1]$$

$$q'_{n,k,r}(h,\delta) \equiv Genq_{n,k,\delta}, \text{ for } h > r \qquad [4.2]$$

and

$$Eq\tau_{r,\delta}(\theta_1) \equiv \left(\frac{1}{B(n-r,\ r+1)} \int_0^{1-\theta_1} t^{n-r-1} (1-t)^r \, dt + \sum_{\substack{n_1 \ge 0,...,n_k \ge 0 \\ \sum_{i=1}^{k} n_i = n \\ \min_{j=2,...,k} n_j \ge \delta}} \frac{n!}{n_1! \ ... \ n_k!} \theta_1^{n_1} \ ... \ \theta_k^{n_k} \right)$$

$$= \sum_{h \ge 0} \binom{n-h}{h} q'_{n,k,r}(h,\delta) \ \theta_1^h \ (1 - \sum_{i=2}^{k} \theta_i)^{n-h} \qquad [4.3]$$

a particular tail, which we might refer to as "δ*-equidistant-fromLevel-r tail*" (as, for $h > r$, the weights $q'_{n,k,r}(h,\delta)$ do not depend on h). In intuitive terms, the above lower tail, where the lower bound on the minimum frequency is constantly equal to δ, can be intuitively imagined as a multinomial lower tail which, "in its upper end" (precisely, for $h > r$), "follows" at distance δ, the "shape" of the border of the sample space $S_{n,k}$. In case $\delta = 0$ it is equal to 1, while, for δ sufficiently large ($\delta > \left[\frac{n-k}{k-1}\right]$), it is coincident with the probability of n_1 being less than or equal to r:

$$prob(n_1 \le r) \le Eq\tau_{r,\delta}(\theta_1) \le 1 \ . \qquad [4.4]$$

As to the weights of this tail, for $h > r$, applying the same argument used in proof of Lemma 4.G we have:

$$q'_{n,k,r}(h) \ge q'_{n,k,r}(h+1) \qquad [4.5]$$

Therefore, we can conclude that also the "δ *-equidistant-fromLevel-r tail*" is minimized by the equal probability configuration. Such result is somehow "more stringent" than the conjecture by Gupta and Nagel, in the sense that the decrease rate of this tail is comprised between the ordinary binomial tail and the tail considered by Gupta and Nagel (cf. Table 10 in section 7).

5. HISTORICAL BACKGROUND ON THE GUPTA-NAGEL CONJECTURE

The two main approaches to *selection and ranking* problems are commonly referred to as the *indifference-zone* approach and the *subset-selection* approach. Comprehensive bibliography and explanations on these two different approaches to selection procedures can be found, for instance, in Gupta and Panchapakesan (1985). Leading research is carried out by P. Chen (cf. for instance, P. Chen (1985), (1986)). For the problem of selection of the least likely event, under the indifference-zone approach, an older reference can be made to Alam and Thompson (1972), while, for the problems of selecting the most probable event under the indifference-zone approach and under the subset-selection approach, Bechhofer, Elmaghrabi, and Morse (1959) and to Gupta and Nagel (1967), can be considered, respectively.

Gupta and Nagel's subset-selection procedure for the least probable multinomial event (or cell) is as follows:

$$\textit{Select the cell with observed } x_i \textit{ iff } x_i \leq \min\{x_1,...,x_n\} + r_\alpha$$

where $n_1,...,n_k$ is the observed sample from a multinomial distribution with cell probabilities $\theta_1, ..., \theta_k$ and r_α is the smallest non-negative integer such that we have a *probability of a correct selection* greater than or equal to a prespecified constant α. Gupta and Nagel (1967, p.1) define a *correct* selection as the selection of any subset of the *k* cells which contains the cell with the smallest probability (also specifying that "in the case of a tie, one of the cells with the smallest value is considered "tagged" and the selection is correct if this "tagged" cell is in the selected subset"). The probability of a correct selection (PCS) of the subset of cells with Gupta-Nagel's selection procedure is given by Gupta and Nagel (1967) as the multinomial tail [1.1], where, in case of ties, i.e., $\theta_1 = ... = \theta_h$, θ_1 denotes the probability corresponding to the "tagged" cell.

In order to carry out Gupta and Nagel's subset selection procedure, it is, therefore, necessary to know the specific configuration $\theta_1^*, ..., \theta_k^*$ which minimizes the PCS, referred to as the *least favorable configuration* (LFC).

In the literature, similar problems are often solved by the following two-step method:

1. Given θ_1 (or θ_k, in problems of selection of the most probable event), find the configuration of the cell-probabilities which minimizes the PCS, 2. Determine the minimizing value of θ_1 (or θ_k). Typically, in step 1 properties of families of distributions parameterized to preserve Schur-convexity are used (see, for instance, Proschan and Sethuraman (1977), pp.1-2), while step 2 is usually carried out through a monotonicity result. In fact, it is common that the probability of a correct selection is a Schur-concave function after some conditioning, and a second step is usually necessary to remove the conditioning (cf. Marshall and Olkin (1979), pp. 396-400).

Such a two-stage method was essentially used, for instance, by Bechhofer, Elmaghrabi, and Morse (1959) and by Alam and Thompson (1972), for the determination of the LFC for selecting the least probable and the most probable multinomial event, respectively, under the indifference-zone approach. Gupta and Nagel (1967) also employed this method to derive the LFC for selecting the least probable multinomial event under the subset-selection approach. In particular, they implicitly proved that the PCS is a Schur-concave function of $\theta_2, ..., \theta_k$ if θ_1 is fixed, so that the LFC for this problem is the so-called "slippage" configuration [3.8]. They also conjectured (1967, p. 9) that the LFC is the equal probability configuration.

6. A STATISTICALLY INTUITIVE ARGUMENT

After having recalled the origin of the problem, it can be interesting to provide also an intuitive argument which resorts to the original meaning, as provided by S. Gupta, of the objective function, and that could be used as an informal justification of the conjecture. Recall the statistical meaning of the problem. We are looking for the *least favorable configuration*, that is the configuration of cell probabilities which minimizes the probability of a correct selection (PCS). An intuitive explanation might consist of two observations.

Observation 1. Since this probability is a continuous function symmetrical with respect to $\theta_2,...,\theta_k$, it is intuitive that a solution has the form $(\theta_1, \frac{1-\theta_1}{k-1}, ..., \frac{1-\theta_1}{k-1})$ (the so called *slippage configuration*). In fact, for each given θ_1, if a configuration $(\theta_1,...,\theta_A,...,\theta_B,...)$ minimizes the PCS, by symmetry, also $(\theta_1,...,\theta_B,...,\theta_A,...)$ does it. This might justify the above slippage configuration (Gupta and Nagel, 1967).

Observation 2. Having guessed that the minimum is reached at the slippage configuration and bearing in mind the symmetry of the probability of making a correct selection, let us now view such a probability as a function of θ_1. Take for instance $r = 0$ (but the observation could be made for any positive r). Being θ_1 the <u>smallest</u> probability and n_1 the corresponding frequency, a correct selection occurs when there are no "transgressions" (to what is most probable to happen, i.e., that n_1 be less than all other frequencies) such as $n_1 > n_2$ or $n_1 > n_3$, and so on. Of course, it should be evident that the probability of such a "transgression" is 0 when $\theta_1 = 0$ (because n_1 will always be 0), and it is intuitive that it will be increasing as θ_1 increases, because the distance between θ_1 and the other probabilities is reduced, and the chance that one of the corresponding frequencies might occur to be less than n_1 becomes more probable. Thus, it should be intuitive that by increasing θ_1, also increases the chance that a "transgression" will occur, and that a wrong selection is made.

7. A SPECIFIC ILLUSTRATION OF THE PROOF

The Reviewers of this paper have kindly requested an illustration of the proof for a specific case, with $n = 20,\ k = 5,\ r = 2$. This section contains such an illustration and aims to provide some further insight on intuitive aspects of the methods employed.

Lemmas 1, 2 are straightforward, Lemma 3 is due to Gupta-Nagel (1967). We shall illustrate Theorem 2, emphasizing its conceptual aspects, and will provide numerical illustration of results in Lemma 4.

Illustration of Theorem 2

Under the known settings, in Theorem 2, we wish to minimize:

$$\tau_2(\theta_1,\ldots,\theta_5) \equiv \sum_{C_{20,5,2}} \frac{20!}{n_1!\ n_2!\ n_3!\ n_4!\ n_5!}\ \theta_1^{n_1}\ \left(\frac{1-\theta_1}{k-1}\right)^{20-n_1}$$

where

$$C_{20,5,2} \equiv \left\{\ (n_1,\ldots,n_5) \in Z^5\ |\ \sum_{j=1}^{5} n_j = 20 \text{ and } n_1 \le \min\{n_2,\ldots,n_5\} + 2 \right\}$$

(this is the subset of samples which lead to a "*correct selection*").

If we view the set $C_{20,5,2}$ as partitioned with respect to the values of n_1, we can write the above sum as a "decomposition into binomial components":

$$\tau_2(\theta_1,\ldots,\theta_5) = \sum_{P_0 \cup P_1 \cup P_2} \frac{20!}{n_1!\ n_2!\ n_3!\ n_4!\ n_5!}\ \theta_1^{n_1}\ \left(\frac{1-\theta_1}{4}\right)^{n-n_1} + \sum_{h=3}^{i_0} \sum_{P_h \cap C_{20,5,2}} \frac{n!}{h!\ \ldots\ n_5!}\ \theta_1^{h}\ \left(\frac{1-\theta_1}{4}\right)^{20-h}$$

where

$$P_h \equiv \left\{\ (n_1,\ldots,n_5) \in Z^5\ |\ \sum_{i=1}^{5} n_i = 20 \text{ and } n_1 = h\ \right\},\quad h = 0,\ldots,20\,.$$

Here we have split the decomposition into two parts. In the first part, the sets P_h are taken entirely because, by Lemma 1, their points are all included in the *correct-selection zone* $C_{20,5,2}$ (cf. Fig. 1), hence the summands will be the ordinary binomial probabilities. In the second sum, the sets P_h are taken partially because not all their configurations are in $C_{20,5,2}$. The $(n_1,\ldots,n_5)$ with $n_1 > i_0$ are not included in the decomposition because, by Lemma 2, we know that they are excluded from the *correct-selection zone* $C_{20,5,2}$ (cf. Fig. 1).

Applying Lemma 2, we find that $i_0 = 5$. Thus, we can write:

$$\tau_2(\theta_1,\ldots,\theta_5) = \sum_{h=0}^{2} \binom{20}{h} \theta_1^h\ (1-\theta_1)^{20-h} + \sum_{h=3}^{5} \left[\frac{1}{4^{20-h}} \sum_{P_h \cap C_{20,5,2}} \frac{(20-h)!}{n_2!\ \ldots\ n_5!} \right] \binom{20}{h} \theta_1^h\ (1-\theta_1)^{20-h}$$

and, denoting by $q_{20,5,2}(h) \equiv \frac{1}{4^{20-h}} \sum_{P_h \cap C_{20,5,2}} \frac{(20-h)!}{n_2!\ \ldots\ n_5!}$ the weights of the binomial terms in the second sum,

$$= \sum_{h=0}^{2} \binom{n}{h} \theta_1^h\ (1-\theta_1)^{n-h} + \sum_{h=3}^{5} q_{n,k,r}(h) \binom{n}{h} \theta_1^h\ (1-\theta_1)^{n-h}$$

for more clarity, let us denote the binomial terms $\theta_1^h \ (1-\theta_1)^{n-h}$ with $BinT_h(\theta_1)$

$$= \sum_{h=0}^{2} BinT_h(\theta_1) + \sum_{h=3}^{5} q_{20,5,2}(h)\ BinT_h(\theta_1)$$

or, considering that for $h = 0,1,2$ we have $q_{20,5,2}(h) = 1$,

$$= \sum_{h=0}^{5} q_{20,5,2}(h)\ BinT_h(\theta_1) \qquad [7.1]$$

where $0 < q_{20,5,2}(h) < 1$, for $h = r+1, \ldots, i_0$, $r < i_0$. In particular, exact computations yield the following values:

$q_{20,5,2}(h) = 1$, for $h = 0,1,2$,
$q_{20,5,2}(3) = 0{,}969977988861501$,
$q_{20,5,2}(4) = 0{,}753139955922961$,
$q_{20,5,2}(5) = 0{,}231012515723705$,
$q_{20,5,2}(h) = 0$ for $h = 6,\ldots,20$,

We wish to show that the sum [7.1], where some summands ($h = 3,4,5$) are weighed, is a decreasing function of $\theta_1 \in (0, \frac{1}{k}]$.

The intuition beyond the proof is as follows. We compare term by term above sum [7.1] with the following sum (ordinary binomial tail):

$$\sum_{h=0}^{5} BinT_h(\theta_1) \qquad [7.2]$$

which we know to be decreasing in $\theta_1 \in (0, \frac{1}{k}]$ (cf. [1.13]).

To do the comparison we split the interval $\theta_1 \in (0, \frac{1}{k}]$ into subintervals, each of which has length $\frac{1}{n}$ and compare the two sums in each of such subintervals.

In a generic interval $(\frac{i}{n}, \frac{i+1}{n}]$ we can note that the first terms ($h = 0,\ldots, i$) $BinT_h(\theta_1)$ are decreasing (have negative derivative) while the last terms ($h = i+1,\ldots, i_0 - r$) $BinT_h(\theta_1)$ are increasing (have positive derivative). The sum [7.2] of all terms (binomial tail), being decreasing, has a negative derivative. In [7.1] we have a similar situation, with the difference that the derivatives are weighted with decreasing weights $q_{20,5,2}(h)$.

Such a weighting, where the negative terms are multiplied by weights each of which is larger than any of the weights applied to the positive terms, will also result in a negative derivative for [7.1] (cf. Figure 2).

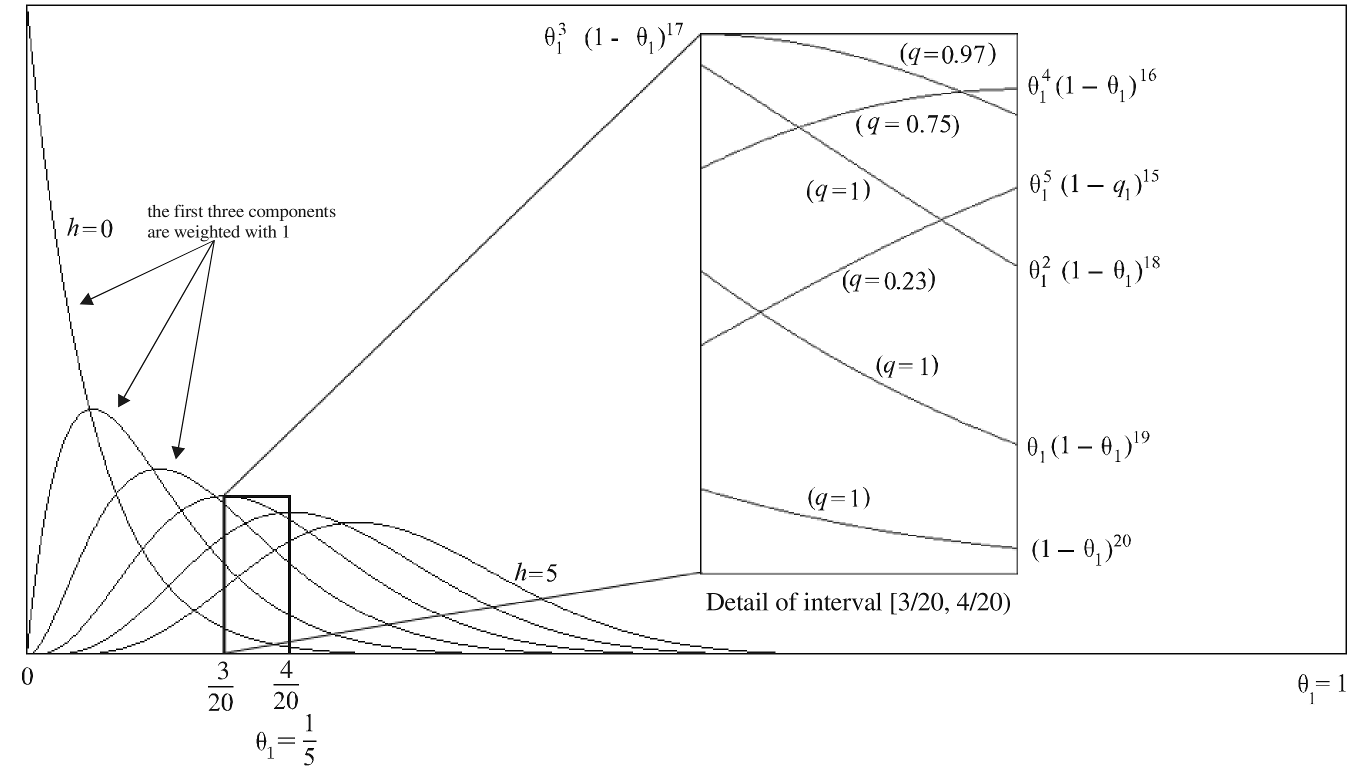

Figure 2.

Binomial terms $\binom{20}{h}\ \theta_1^h\ \ (1-\theta_1)^{20-h}$ for different values of *h*. In the top-right frame, what happens in the interval $\theta_1 \in (\frac{3}{20},\frac{4}{20}]$ is shown in detail. For each binomial term, the corresponding weight $q_{20,5,2}(h)$ is also indicated within parentheses.

Numerical illustration of results in Lemma 4

Lemma 4 is illustrated by showing that, in the case suggested by the Reviewers, the various relationships hold.

Summary tables of computations ($n = 20,\ k = 5,\ r = 2$)

Table 1. Illustration of [3.27]-[3.33]

h	$SumC_{n,k+1,r} \equiv k^{n-h}\ q_{n,k+1,r}(h)$	$SumW_{n,k+1,r} \equiv k^{n-h}\ (1-q_{n,k+1,r}(h))$	k^{n-h}	$q_{n,k+1,r}(h)$	$1-q_{n,k+1,r}(h)$
0	95,367,431,640,625	0	95,367,431,640,625	1	0
1	19,073,486,328,125	0	19,073,486,328,125	1	0
2	3,814,697,265,625	0	3,814,697,265,625	1	0
3	678,330,198,120	84,609,255,005	762,939,453,125	0.889100957279846	0.110899042720154
4	61,305,644,400	91,282,246,225	152,587,890,625	0.40177267113984	0.59822732886016
5	168,168,000	30,349,410,125	30,517,578,125	0.005510529024	0.994489470976
6	0	6,103,515,625	6,103,515,625	0	1
7	0	1,220,703,125	1,220,703,125	0	1
8	0	244,140,625	244,140,625	0	1
9	0	48,828,125	48,828,125	0	1
10	0	9,765,625	9,765,625	0	1
11	0	1,953,125	1,953,125	0	1
12	0	390,625	390,625	0	1
13	0	78,125	78,125	0	1
14	0	15,625	15,625	0	1
15	0	3,125	3,125	0	1
16	0	625	625	0	1
17	0	125	125	0	1
18	0	25	25	0	1
19	0	5	5	0	1
20	0	1	1	0	1

Table 2. ***Illustration of*** **[3.32]-[3.37]**

h	$SumW1_{n,k+1,r}$	$SumW2_{n,k+1,r}$	$(SumW1_{n,k+1,r} + SumW2_{n,k+1,r})$	$k^{n-h} - (SumW1_{n,k+1,r} + SumW2_{n,k+1,r})$
0	0	0	0	95,367,431,640,625
1	0	0	0	19,073,486,328,125
2	0	0	0	3,814,697,265,625
3	67,945,160,045	16,664,094,960	84,609,255,005	678,330,198,120
4	76,162,381,465	15,119,864,760	91,282,246,225	61,305,644,400
5	29,016,678,725	1,332,731,400	30,349,410,125	168,168,000
6	6,103,515,625	0	6,103,515,625	0
7	1,220,703,125	0	1,220,703,125	0
8	244,140,625	0	244,140,625	0
9	48,828,125	0	48,828,125	0
10	9,765,625	0	9,765,625	0
11	1,953,125	0	1,953,125	0
12	390,625	0	390,625	0
13	78,125	0	78,125	0
14	15,625	0	15,625	0
15	3,125	0	3,125	0
16	625	0	625	0
17	125	0	125	0
18	25	0	25	0
19	5	0	5	0
20	1	0	1	0

Tables 3-4. ***Illustration of*** **[3.49]** ***for various h***

$h = 0$

u	$\binom{n-h}{u}$	$(k-1)^{n-u-h}$	$q_{n-u,k,r}(h)$	$\binom{n-h}{u}(k-1)^{n-u-h}\, q_{n-u,k,r}(h)$
0	1	1,099,511,627,776	1	1,099,511,627,776
1	20	274,877,906,944	1	5,497,558,138,880
2	190	68,719,476,736	1	13,056,700,579,840
3	1,140	17,179,869,184	1	19,585,050,869,760
4	4,845	4,294,967,296	1	20,809,116,549,120
5	15,504	1,073,741,824	1	16,647,293,239,296
6	38,760	268,435,456	1	10,404,558,274,560
7	77,520	67,108,864	1	5,202,279,137,280
8	125,970	16,777,216	1	2,113,425,899,520
9	167,960	4,194,304	1	704,475,299,840
10	184,756	1,048,576	1	193,730,707,456
11	167,960	262,144	1	44,029,706,240
12	125,970	65,536	1	8,255,569,920
13	77,520	16,384	1	1,270,087,680
14	38,760	4,096	1	158,760,960
15	15,504	1,024	1	15,876,096
16	4,845	256	1	1,240,320
17	1,140	64	1	72,960
18	190	16	1	3,040
19	20	4	1	80
20	1	1	1	1
			Tot	95,367,431,640,625

$h = 1,2,3 \quad [omissis]$

$h = 4$

u	$\binom{n-h}{u}$	$(k-1)^{n-u-h}$	$q_{n-u,k,r}(h)$	$\binom{n-h}{u}(k-1)^{n-u-h}\, q_{n-u,k,r}(h)$
2	120	268,435,456	0.61824095249176	19,914,935,040
3	560	67,108,864	0.531864166259766	19,987,968,000
4	1,820	16,777,216	0.433437824249268	13,234,821,600
5	4,368	4,194,304	0.326042175292969	5,973,327,360
6	8,008	1,048,576	0.216293334960938	345,945,600
7	11,440	262,144	0.1153564453125	32,432,400
8	12,870	65,536	0.0384521484375	0
9	11,440	16,384	0	0
10	8,008	4,096	0	0
11	4,368	1,024	0	0
12	1,820	256	0	0
13	560	64	0	0
14	120	16	0	0
15	16	4	0	0
16	1	1	0	0
			Tot	61,305,644,400

$h = 6{,}7, \ldots\ [omissis]$

Tables 5-6. Illustration of the order transition formulas [3.52]-[3-57]

We set, for instance, $n = 20,\ k = 5, r = 2,\ sum = n,\ bound = h - r$

h	$Genq_{n+1,k,sum+1,bound}(h+1)$	$Genq_{n,k,sum,bound}(h)$
0-2	1	1
3	0.969977988861501	0.969977988861501
4	0.753139955922961	0.753139955922961
5	0.231012515723705	0.231012515723705
6-20	0	0

Table 7. Illustration of proof of Lemma4.G (*Weight Sequence monotonicity*)

h	$q_{n,k,r}(h)$ $= Genq_{sum,k,bound}(h)$	$q_{n,k,r}(h+1)$	$Genq_{sum-1,k,bound}(h-1)$	$Genq_{sum-1,k,bound-1}(h-1)$
0	1	1	1	1
1	1	1	1	1
2	1	0,969977988861501	0,969977988861501	0,960001168772578
3	0,969977988861501	0,753139955922961	0,753139955922961	0,691806972026825
4	0,753139955922961	0,231012515723705	0,231012515723705	0,144089162349701
5	0,231012515723705	0	0	0
6-20	0	0	0	0

Finally, let us disclose the intuition beyond the usage of the generalized weights to establish the monotonicity of the sequence of binomial weights in the decomposition into binomial components. Consider the difference between consecutive weights and take, for instance, $h = 3$

$$q_{n,k,r}(h+1 := 4) - q_{n,k,r}(h := 3)$$

$$= \frac{1}{4^{n-4}} \sum_{\substack{n_2,\ldots,n_k \\ \sum_{i=2}^{k} n_i = 20-4 \\ 4 \le \min_{j=2,\ldots,k} n_j + r}} \frac{(20-4)!}{n_2!\ \ldots\ n_k!} - \frac{1}{(k-1)^{20-3}} \sum_{\substack{n_2,\ldots,n_k \\ \sum_{i=2}^{k} n_i = 20-3 \\ 3 \le \min_{j=2,\ldots,k} n_j + r}} \frac{(20-3)!}{n_2!\ \ldots\ n_k!}$$

In this specific case, the weight $q_{20,5,2}(3)$ is a sum of 560 terms. The weight $q_{20,5,2}(4)$ is a sum of 165 terms. These terms are listed in Table 1 (those relative to $q_{20,5,2}(3)$ are on the left of each column and those relative to $q_{20,5,2}(4)$ on the right, arranged as to show the correspondence).

Table 8. Correspondence between set of partitions

The 560 configurations making up $q_{20,5,2}(3)$ *(on the left of each column) and the 165 configurations making up* $q_{20,5,2}(4)$.

1 1 1 14	1 5 5 6		2 1 1 13	2 6 6 3		3 5 2 7	2 5 2 7	4 5 1 7		5 6 5 1		7 3 1 6	
1 1 10 5	1 5 6 5		2 1 10 4	2 6 7 2		3 5 3 6	2 5 3 6	4 5 2 6	3 5 2 6	5 7 1 4		7 3 2 5	6 3 2 5
1 1 11 4	1 5 7 4		2 1 11 3	2 6 8 1		3 5 4 5	2 5 4 5	4 5 3 5	3 5 3 5	5 7 2 3	4 7 2 3	7 3 3 4	6 3 3 4
1 1 12 3	1 5 8 3		2 1 12 2	2 7 1 7		3 5 5 4	2 5 5 4	4 5 4 4	3 5 4 4	5 7 3 2	4 7 3 2	7 3 4 3	6 3 4 3
1 1 13 2	1 5 9 2		2 1 13 1	2 7 2 6		3 5 6 3	2 5 6 3	4 5 5 3	3 5 5 3	5 7 4 1		7 3 5 2	6 3 5 2
1 1 14 1	1 6 1 9		2 1 2 12	2 7 3 5		3 5 7 2	2 5 7 2	4 5 6 2	3 5 6 2	5 8 1 3		7 3 6 1	
1 1 2 13	1 6 2 8		2 1 3 11	2 7 4 4		3 5 8 1		4 5 7 1		5 8 2 2	4 8 2 2	7 4 1 5	
1 1 3 12	1 6 3 7		2 1 4 10	2 7 5 3		3 6 1 7		4 6 1 6		5 8 3 1		7 4 2 4	6 4 2 4
1 1 4 11	1 6 4 6		2 1 5 9	2 7 6 2		3 6 2 6	2 6 2 6	4 6 2 5	3 6 2 5	5 9 1 2		7 4 3 3	6 4 3 3
1 1 5 10	1 6 5 5		2 1 6 8	2 7 7 1		3 6 3 5	2 6 3 5	4 6 3 4	3 6 3 4	5 9 2 1		7 4 4 2	6 4 4 2
1 1 6 9	1 6 6 4		2 1 7 7	2 8 1 6		3 6 4 4	2 6 4 4	4 6 4 3	3 6 4 3	6 1 1 9		7 4 5 1	
1 1 7 8	1 6 7 3		2 1 8 6	2 8 2 5		3 6 5 3	2 6 5 3	4 6 5 2	3 6 5 2	6 1 2 8		7 5 1 4	
1 1 8 7	1 6 8 2		2 1 9 5	2 8 3 4		3 6 6 2	2 6 6 2	4 6 6 1		6 1 3 7		7 5 2 3	6 5 2 3
1 1 9 6	1 6 9 1		2 10 1 4	2 8 4 3		3 6 7 1		4 7 1 5		6 1 4 6		7 5 3 2	6 5 3 2
1 10 1 5	1 7 1 8		2 10 2 3	2 8 5 2		3 7 1 6		4 7 2 4	3 7 2 4	6 1 5 5		7 5 4 1	
1 10 2 4	1 7 2 7		2 10 3 2	2 8 6 1		3 7 2 5	2 7 2 5	4 7 3 3	3 7 3 3	6 1 6 4		7 6 1 3	
1 10 3 3	1 7 3 6		2 10 4 1	2 9 1 5		3 7 3 4	2 7 3 4	4 7 4 2	3 7 4 2	6 1 7 3		7 6 2 2	6 6 2 2
1 10 4 2	1 7 4 5		2 11 1 3	2 9 2 4		3 7 4 3	2 7 4 3	4 7 5 1		6 1 8 2		7 6 3 1	
1 10 5 1	1 7 5 4		2 11 2 2	2 9 3 3		3 7 5 2	2 7 5 2	4 8 1 4		6 1 9 1		7 7 1 2	
1 11 1 4	1 7 6 3		2 11 3 1	2 9 4 2		3 7 6 1		4 8 2 3	3 8 2 3	6 2 1 8		7 7 2 1	
1 11 2 3	1 7 7 2		2 12 1 2	2 9 5 1		3 8 1 5		4 8 3 2	3 8 3 2	6 2 2 7	5 2 2 7	7 8 1 1	
1 11 3 2	1 7 8 1		2 12 2 1	3 1 1 12		3 8 2 4	2 8 2 4	4 8 4 1		6 2 3 6	5 2 3 6	8 1 1 7	
1 11 4 1	1 8 1 7		2 13 1 1	3 1 10 3		3 8 3 3	2 8 3 3	4 9 1 3		6 2 4 5	5 2 4 5	8 1 2 6	
1 12 1 3	1 8 2 6		2 2 1 12	3 1 11 2		3 8 4 2	2 8 4 2	4 9 2 2	3 9 2 2	6 2 5 4	5 2 5 4	8 1 3 5	
1 12 2 2	1 8 3 5		2 2 10 3	3 1 12 1		3 8 5 1		4 9 3 1		6 2 6 3	5 2 6 3	8 1 4 4	
1 12 3 1	1 8 4 4		2 2 11 2	3 1 2 11		3 9 1 4		5 1 1 10		6 2 7 2	5 2 7 2	8 1 5 3	
1 13 1 2	1 8 5 3		2 2 12 1	3 1 3 10		3 9 2 3	2 9 2 3	5 1 10 1		6 2 8 1		8 1 6 2	
1 13 2 1	1 8 6 2		2 2 2 11	3 1 4 9		3 9 3 2	2 9 3 2	5 1 2 9		6 3 1 7		8 1 7 1	
1 14 1 1	1 8 7 1		2 2 3 10	3 1 5 8		3 9 4 1		5 1 3 8		6 3 2 6	5 3 2 6	8 2 1 6	
1 2 1 13	1 9 1 6		2 2 4 9	3 1 6 7		4 1 1 11		5 1 4 7		6 3 3 5	5 3 3 5	8 2 2 5	7 2 2 5
1 2 10 4	1 9 2 5		2 2 5 8	3 1 7 6		4 1 10 2		5 1 5 6		6 3 4 4	5 3 4 4	8 2 3 4	7 2 3 4
1 2 11 3	1 9 3 4		2 2 6 7	3 1 8 5		4 1 11 1		5 1 6 5		6 3 5 3	5 3 5 3	8 2 4 3	7 2 4 3
1 2 12 2	1 9 4 3		2 2 7 6	3 1 9 4		4 1 2 10		5 1 7 4		6 3 6 2	5 3 6 2	8 2 5 2	7 2 5 2
1 2 13 1	1 9 5 2		2 2 8 5	3 10 1 3		4 1 3 9		5 1 8 3		6 3 7 1		8 2 6 1	
1 2 2 12	1 9 6 1		2 2 9 4	3 10 2 2	2 10 2 2	4 1 4 8		5 1 9 2		6 4 1 6		8 3 1 5	
1 2 3 11	10 1 1 5		2 3 1 11	3 10 3 1		4 1 5 7		5 10 1 1		6 4 2 5	5 4 2 5	8 3 2 4	7 3 2 4
1 2 4 10	10 1 2 4		2 3 10 2	3 11 1 2		4 1 6 6		5 2 1 9		6 4 3 4	5 4 3 4	8 3 3 3	7 3 3 3
1 2 5 9	10 1 3 3		2 3 11 1	3 11 2 1		4 1 7 5		5 2 2 8	4 2 2 8	6 4 4 3	5 4 4 3	8 3 4 2	7 3 4 2
1 2 6 8	10 1 4 2		2 3 2 10	3 12 1 1		4 1 8 4		5 2 3 7	4 2 3 7	6 4 5 2	5 4 5 2	8 3 5 1	
1 2 7 7	10 1 5 1		2 3 3 9	3 2 1 11		4 1 9 3		5 2 4 6	4 2 4 6	6 4 6 1		8 4 1 4	
1 2 8 6	10 2 1 4		2 3 4 8	3 2 10 2	2 2 10 2	4 10 1 2		5 2 5 5	4 2 5 5	6 5 1 5		8 4 2 3	7 4 2 3
1 2 9 5	10 2 2 3	9 2 2 3	2 3 5 7	3 2 11 1		4 10 2 1		5 2 6 4	4 2 6 4	6 5 2 4	5 5 2 4	8 4 3 2	7 4 3 2
1 3 1 12	10 2 3 2	9 2 3 2	2 3 6 6	3 2 2 10	2 2 2 10	4 11 1 1		5 2 7 3	4 2 7 3	6 5 3 3	5 5 3 3	8 4 4 1	
1 3 10 3	10 2 4 1		2 3 7 5	3 2 3 9	2 2 3 9	4 2 1 10		5 2 8 2	4 2 8 2	6 5 4 2	5 5 4 2	8 5 1 3	
1 3 11 2	10 3 1 3		2 3 8 4	3 2 4 8	2 2 4 8	4 2 10 1		5 2 9 1		6 5 5 1		8 5 2 2	7 5 2 2
1 3 12 1	10 3 2 2	9 3 2 2	2 3 9 3	3 2 5 7	2 2 5 7	4 2 2 9	3 2 2 9	5 3 1 8		6 6 1 4		8 5 3 1	
1 3 2 11	10 3 3 1		2 4 1 10	3 2 6 6	2 2 6 6	4 2 3 8	3 2 3 8	5 3 2 7	4 3 2 7	6 6 2 3	5 6 2 3	8 6 1 2	
1 3 3 10	10 4 1 2		2 4 10 1	3 2 7 5	2 2 7 5	4 2 4 7	3 2 4 7	5 3 3 6	4 3 3 6	6 6 3 2	5 6 3 2	8 6 2 1	
1 3 4 9	10 4 2 1		2 4 2 9	3 2 8 4	2 2 8 4	4 2 5 6	3 2 5 6	5 3 4 5	4 3 4 5	6 6 4 1		8 7 1 1	
1 3 5 8	10 5 1 1		2 4 3 8	3 2 9 3	2 2 9 3	4 2 6 5	3 2 6 5	5 3 5 4	4 3 5 4	6 7 1 3		9 1 1 6	
1 3 6 7	11 1 1 4		2 4 4 7	3 3 1 10		4 2 7 4	3 2 7 4	5 3 6 3	4 3 6 3	6 7 2 2	5 7 2 2	9 1 2 5	
1 3 7 6	11 1 2 3		2 4 5 6	3 3 10 1		4 2 8 3	3 2 8 3	5 3 7 2	4 3 7 2	6 7 3 1		9 1 3 4	
1 3 8 5	11 1 3 2		2 4 6 5	3 3 2 9	2 3 2 9	4 2 9 2	3 2 9 2	5 3 8 1		6 8 1 2		9 1 4 3	
1 3 9 4	11 1 4 1		2 4 7 4	3 3 3 8	2 3 3 8	4 3 1 9		5 4 1 7		6 8 2 1		9 1 5 2	
1 4 1 11	11 2 1 3		2 4 8 3	3 3 4 7	2 3 4 7	4 3 2 8	3 3 2 8	5 4 2 6	4 4 2 6	6 9 1 1		9 1 6 1	
1 4 10 2	11 2 2 2	10 2 2 2	2 4 9 2	3 3 5 6	2 3 5 6	4 3 3 7	3 3 3 7	5 4 3 5	4 4 3 5	7 1 1 8		9 2 1 5	
1 4 11 1	11 2 3 1		2 5 1 9	3 3 6 5	2 3 6 5	4 3 4 6	3 3 4 6	5 4 4 4	4 4 4 4	7 1 2 7		9 2 2 4	8 2 2 4
1 4 2 10	11 3 1 2		2 5 2 8	3 3 7 4	2 3 7 4	4 3 5 5	3 3 5 5	5 4 5 3	4 4 5 3	7 1 3 6		9 2 3 3	8 2 3 3
1 4 3 9	11 3 2 1		2 5 3 7	3 3 8 3	2 3 8 3	4 3 6 4	3 3 6 4	5 4 6 2	4 4 6 2	7 1 4 5		9 2 4 2	8 2 4 2
1 4 4 8	11 4 1 1		2 5 4 6	3 3 9 2	2 3 9 2	4 3 7 3	3 3 7 3	5 4 7 1		7 1 5 4		9 2 5 1	
1 4 5 7	12 1 1 3		2 5 5 5	3 4 1 9		4 3 8 2	3 3 8 2	5 5 1 6		7 1 6 3		9 3 1 4	
1 4 6 6	12 1 2 2		2 5 6 4	3 4 2 8	2 4 2 8	4 3 9 1		5 5 2 5	4 5 2 5	7 1 7 2		9 3 2 3	8 3 2 3
1 4 7 5	12 1 3 1		2 5 7 3	3 4 3 7	2 4 3 7	4 4 1 8		5 5 3 4	4 5 3 4	7 1 8 1		9 3 3 2	8 3 3 2
1 4 8 4	12 2 1 2		2 5 8 2	3 4 4 6	2 4 4 6	4 4 2 7	3 4 2 7	5 5 4 3	4 5 4 3	7 2 1 7		9 3 4 1	
1 4 9 3	12 2 2 1		2 5 9 1	3 4 5 5	2 4 5 5	4 4 3 6	3 4 3 6	5 5 5 2	4 5 5 2	7 2 2 6	6 2 2 6	9 4 1 3	
1 5 1 10	12 3 1 1		2 6 1 8	3 4 6 4	2 4 6 4	4 4 4 5	3 4 4 5	5 5 6 1		7 2 3 5	6 2 3 5	9 4 2 2	8 4 2 2
1 5 10 1	13 1 1 2		2 6 2 7	3 4 7 3	2 4 7 3	4 4 5 4	3 4 5 4	5 6 1 5		7 2 4 4	6 2 4 4	9 4 3 1	
1 5 2 9	13 1 2 1		2 6 3 6	3 4 8 2	2 4 8 2	4 4 6 3	3 4 6 3	5 6 2 4	4 6 2 4	7 2 5 3	6 2 5 3	9 5 1 2	
1 5 3 8	13 2 1 1		2 6 4 5	3 4 9 1		4 4 7 2	3 4 7 2	5 6 3 3	4 6 3 3	7 2 6 2	6 2 6 2	9 5 2 1	
1 5 4 7	14 1 1 1		2 6 5 4	3 5 1 8		4 4 8 1		5 6 4 2	4 6 4 2	7 2 7 1		9 6 1 1	

As apparent from the table, a subset of the 560 configurations of frequencies which make up the sum in $q_{n,k,r}(h)$ is in a one-to-one correspondence with the 165 configurations of frequencies which make up the sum in $q_{n,k,r}(h+1)$.

The method for obtaining the inequality $q_{n,k,r}(h) - q_{n,k,r}(h+1) > 0$ consists of increasing the terms in the sum in $q_{n,k,r}(h+1)$ by a less restrictive condition (the set of configurations of frequencies such that ($\min_{j=2,\ldots,k} n_j \geq (h+1) - r - 2$) is larger than the one where ($\min_{j=2,\ldots,k} n_j \geq (h+1) - r$)), which increases the terms to 969. Again, a subset of these 969 terms will be in correspondence one-to-one with similar terms in the set of 560 configurations. The correspondence consisting in having the same frequencies $(n_2,\ldots,n_k)$ except one (for instance n_2) which is shifted by 1. The relative sum of central multinomial coefficients computed over such a subset is equal to $q_{n,k,r}(h)$, which implies the inequality.

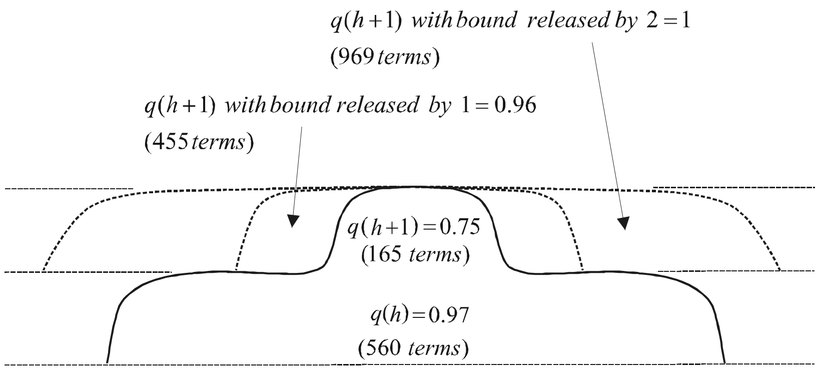

Figure 3.
Symbolic representation of the procedure used to compare two consecutive weights

Table 9. Illustration of proof of Lemma4.G (*Weight Sequence monotonicity*) and Remark 3.5

h	$q_{n,k,r}(h) \equiv \frac{1}{(k-1)^{sum-h}} \sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = sum,\ n_1 = h \\ \min_{j=2,\dots,k} n_j \geq bound}} \frac{(sum-h)!}{n_2!\ \dots\ n_u!\ \dots\ n_k!}$
0	1
1	1
2	1
3	0,969977988861501
4	0,753139955922961
5	0,231012515723705
6-20	0

h	$\frac{1}{(k-1)^{sum-(h+1)}} \sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = sum,\ n_1 = h \\ \min_{j=2,\dots,k} n_j \geq bound}} \frac{(sum-(h+1))!}{n_2!\ \dots\ (n_u-1)!\ \dots\ n_k!}$ (*sum over the "forward image"*)	$\frac{1}{(k-1)^{sum-(h-1)}} \sum_{\substack{n_2,\dots,n_k \\ \sum_{j=1}^{k} n_j = sum,\ n_1 = h \\ \min_{j=2,\dots,k} n_j \geq bound}} \frac{(sum-(h-1))!}{n_2!\ \dots\ (n_u+1)!\ \dots\ n_k!}$ (*sum over the "backward image"*)
0	1	0,9976215910458
1	1	0,996828788061066
2	1	0,995771717414755
3	0,969977988861501	0,94374878302915
4	0,753139955922961	0,696503243409097
5	0,231012515723705	0,200052363798022
6-20	0	0

Table 10. Illustration of results in discussion section 4 ($n = 20,\ k = 5,\ r = 2,\ \delta = 1$)

h	$q_{n,k,r}(h)$	$q^{'}_{n,k,r}(h,\delta)$
0	1	1
1	1	1
2	1	1
3	0,969977988861501	0,969977988861501
4	0,753139955922961	0,960001168772578
5	0,231012515723705	0,946729257702827
6	0	0,929094403982162
7	0	0,90570330619812
8	0	0,874759197235107
9	0	0,833988189697266
10	0	0,780601501464844
11	0	0,71136474609375
12	0	0,6229248046875
13	0	0,5126953125
14	0	0,380859375
15	0	0,234375
16	0	0,09375
17-20	0	0

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